# Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model

$H + M = \text{Lopes}^2$


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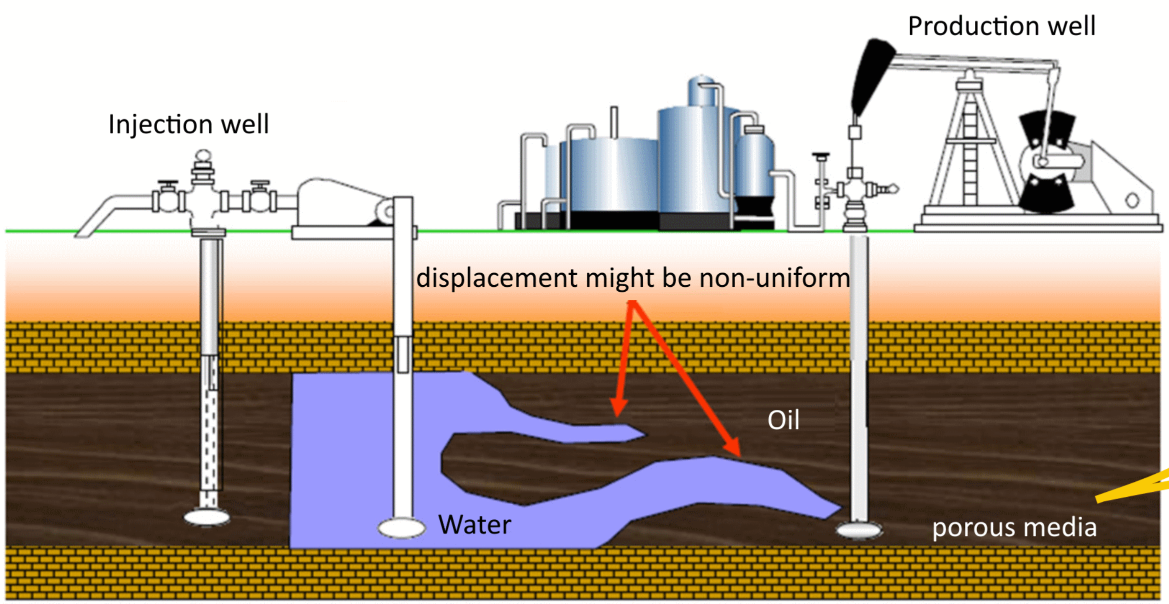
The talk is based on:

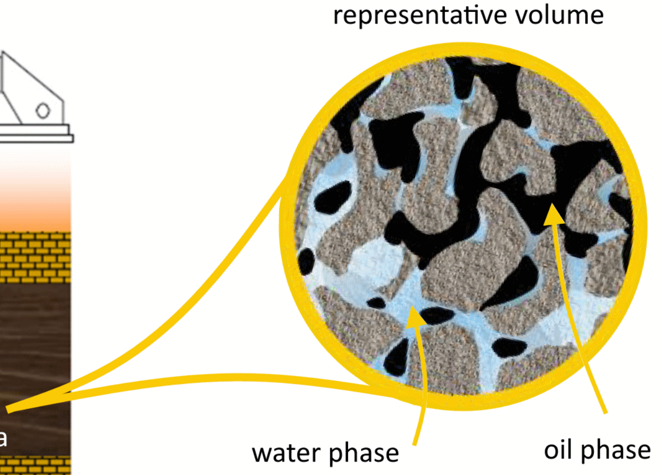
- **Y. Petrova**, B. Plohr, and Marchesin D. Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model. arXiv:2211.10326.
- F. Bakharev, A. Enin, **Y. Petrova**, and N. Rastegaev. Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. Journal of Hyperbolic Differential Equations, 20:1–26, 2023.

# Motivation: enhanced oil recovery (EOR)

We are interested in the mathematical model of oil recovery.

- porous media (averaged models of flow)
- unknown variables: $s \in [0,1]$ - water saturation, $1 - s$ - oil concentration
- relatively small speeds ($\approx 1$ meter per day): Navier-Stokes $\rightarrow$ Darcy's law
- multiphase flow: oil, water, gas
- applications to EOR methods: thermal, gas, chemical flooding



# Two main directions of investigation

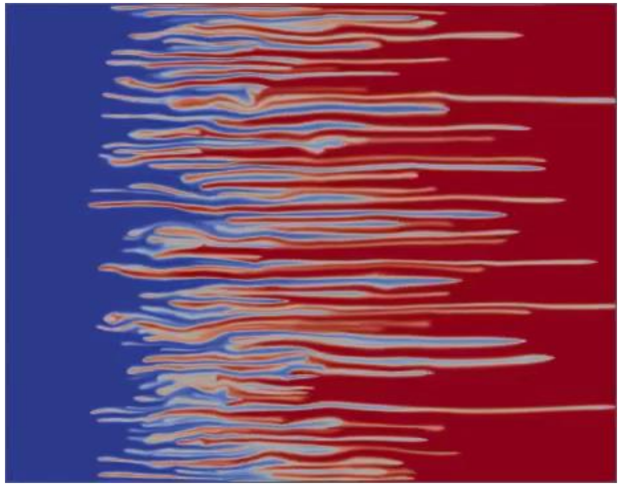
*Stable displacement* (1-dim)

- hyperbolic conservation laws
- main question: find an exact solution for a Riemann problem

$$s_t + \quad f(s,c)_x = 0$$
$$(sc)_t + \big(cf(s,c)\big)_x = 0$$

Example: polymer model

*Unstable displacement* (2-dim)



- viscous fingering phenomenon
- source of instability: water and oil/polymer have different viscosities

$$c_t + div(uc) = \Delta c$$
$$div(u) = 0, \qquad u = -m(c)\,\nabla p$$

Example: incompressible porous media equation (IPM)
[remember yesterday's talk of Sergey Tikhomirov]

# Glimm-Isaacson model (KKIT)*

Two-phase oil-water flow with *polymer* in the water (1980)

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (cs)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$

- $s \in [0,1]$ – water saturation
- $c \in [0,1]$ – polymer concentration in water
- $f(s,c)$ – fractional flow function: affected by polymer
  - S-shaped in $s$ (for fixed $c$)

Case 1:
$f_c' < 0$ (monotone in $c$)

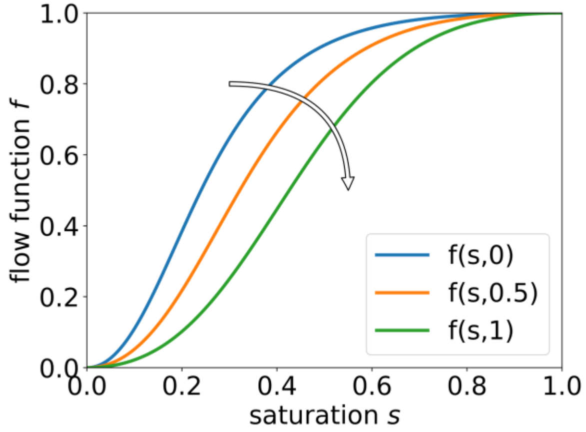
Case 2:
$f$ changes monotonicity once

Initial data: $$(s,c)(x,0) = \begin{cases} (s_L, c_L), & x \le 0 \\ (s_R, c_R), & x \ge 0 \end{cases}$$

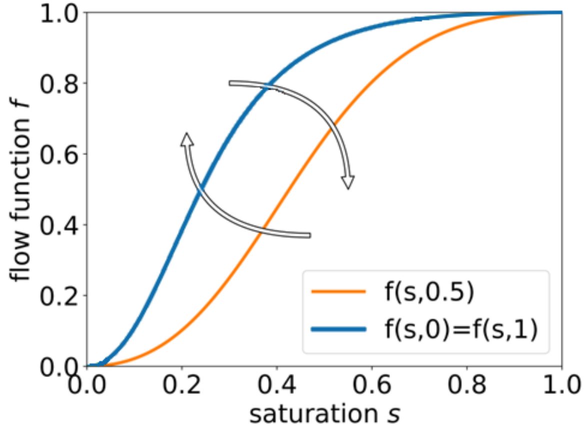
Question: find an exact solution $s(x,t)$ and $c(x,t)$ to any Riemann problem

NB: $s(x,t) = s(x/t)$ – self-similar

Case 1:




Case 2:




* KKIT = Keyfitz, Kranzer, Isaacson, Temple

# Main idea

Polymer model (1980' E. Isaacson)

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_0)$$

- Contact discontinuities (linearly degenerate) ⇒ non-uniqueness of solutions to a Riemann problem
- Vanishing viscosity criteria doesn't help
- Existing admissibility criteria needs to be justified from "physical" perspective

Aim: select unique physically admissible weak solution

Main idea: add small physical effect – adsorption of polymer on the rock (1987 ' T. Johansen, R. Winther)

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_\alpha)$$

Here $\alpha > 0$ - small, $a(c)$ – strictly convex

- Contact discontinuities ⇝ rarefaction and shock waves ⇝ vanishing viscosity criteria is applicable

*Vanishing adsorption criterion*: the admissible contacts for $M_0$ are the $L^1_{loc}$ limits of a family of admissible solutions for a Riemann problem for $M_\alpha$ as $\alpha \to 0$.

# Hyperbolic systems of conservation laws*

$$U_t + F(U)_x = 0$$
$$U_t + DF(U) \cdot U_x = 0$$
Eigenvalues of $DF(U)$ play crucial role

$$\boxed{G(U)_t + F(U)_x = 0}$$

- $G(U) \in \mathbb{R}^n$ - accumulation function (conserved quantities)
- $F(U) \in \mathbb{R}^n$ - flux function (flux of conserved quantities)

Simplest example: wave equation

$$y_{tt} - c^2 y_{xx} = 0$$

(J. d'Alambert, 1750)

can be rewritten as a system of two first-order equations on the state-vector $u = \begin{pmatrix} y_x \\ y_t \end{pmatrix}$

$$u_t + Du_x = 0 \qquad \text{with} \qquad D = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

- eigenvalues $\lambda_1 = c$ and $\lambda_2 = -c$ are real, the system is hyperbolic (strictly).
  Solution contains two wave modes that propagate at the velocities $\lambda_1$ and $\lambda_2$.

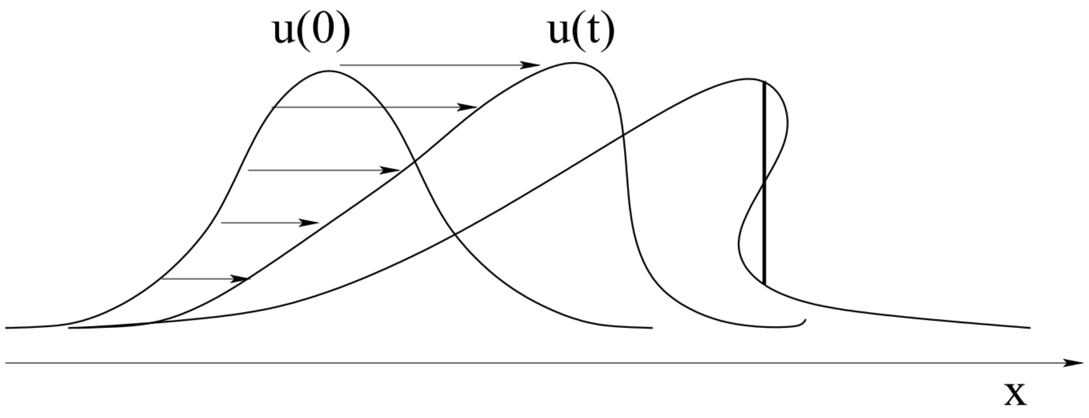
*A. Bressan "Hyperbolic conservation laws: an illustrated tutorial" , lecture notes from Cetraro, Italy 2009

# Hyperbolic systems of conservation laws

- Inviscid Burgers equation (1948)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

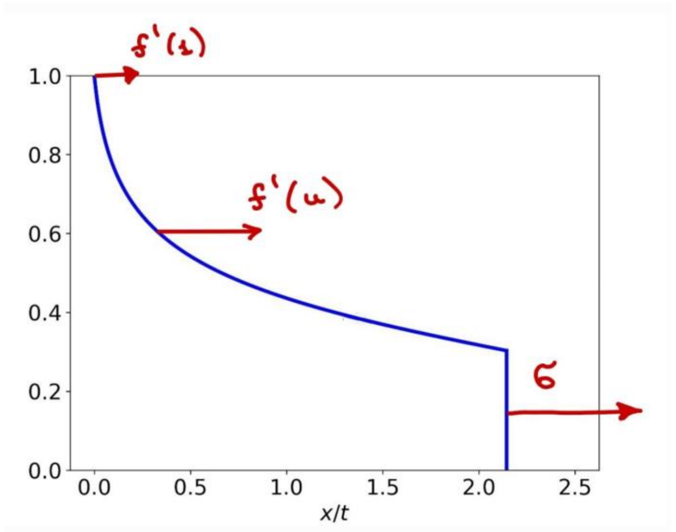
- non-linearity implies wave speed $\lambda(u) = u$ depends on state $u$
- so the wave can spread (rarefaction wave) or concentrate (shock wave)




- General scalar conservation law

$$u_t + f(u)_x = 0$$

- existence, uniqueness of solution of Cauchy problem was established by Olga Oleinik (1957)




Shock speed $\sigma$ between the states $u_-$ and $u_+$ is defined by the Rankine-Hugoniot condition $\sigma = \dfrac{f(u_-) - f(u_+)}{u_- - u_+}$

# Riemann problem (1858)

Riemann solved the initial-value problem with data having a single jump

$$U(x,0) = \begin{cases} U_L, & x \le 0 \\ U_R, & x \ge 0 \end{cases}$$

Solution to a Riemann problem is important because:

- it appears in a long-term behaviour of Cauchy problem
- helps to prove the existence of solutions to Cauchy problem (Glimm's method)
- helps to construct numerical solution (Godunov method)

$$\boxed{U_t + F(U)_x = 0}$$

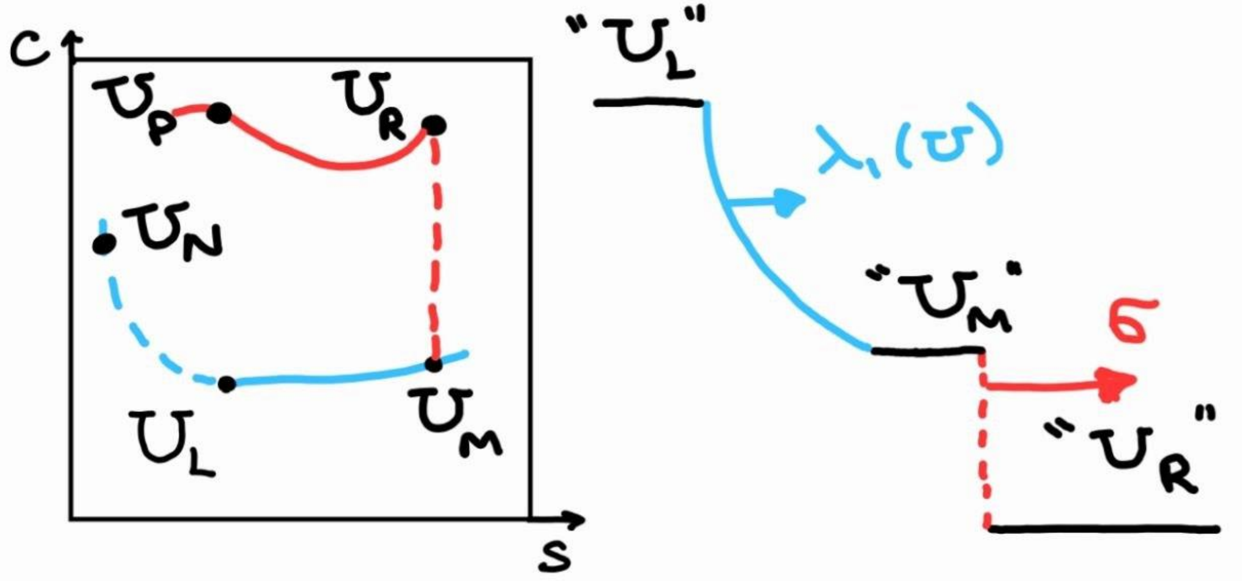
Representation of solution $U = (s, c)$ for $2 \times 2$ systems:

Eigenvalues of $DF(U)$: $\lambda_1(U) < \lambda_2(U)$

Take $U_L$ and $\lambda_1(U)$

Take $U_R$ and $\lambda_2(U)$

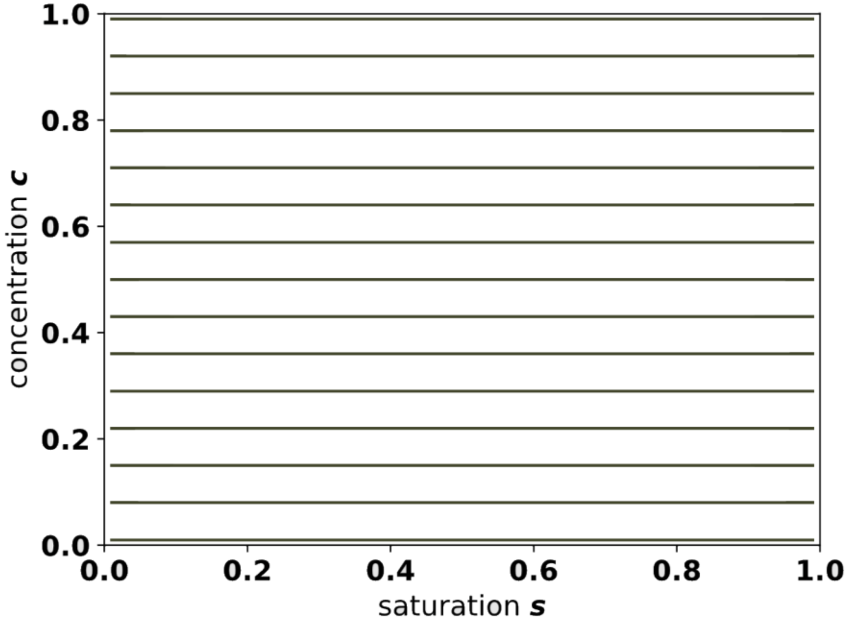
We have constructed a solution!



# Characteristic families: s and c-waves

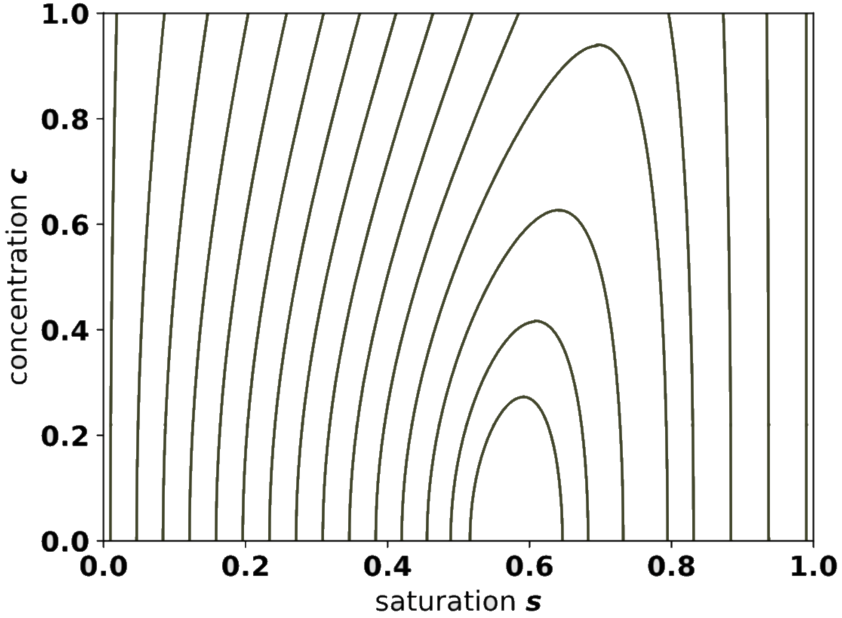
## $s$-waves

- $\lambda^s = f_s'$
- Solve the Buckley-Leverett equation $c = const$
- Riemann invariant $c = const$
- "line" family



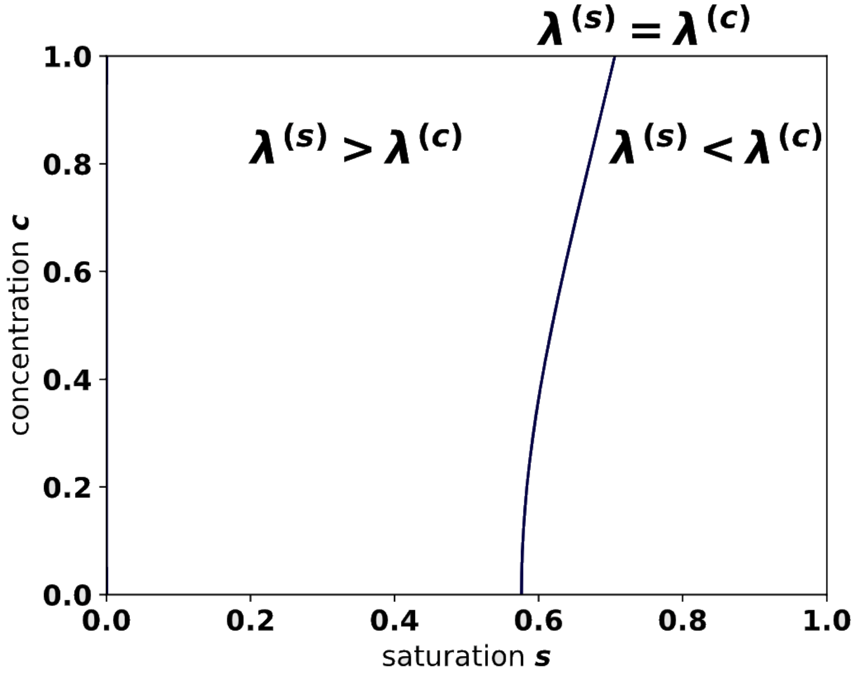

## $c$-waves

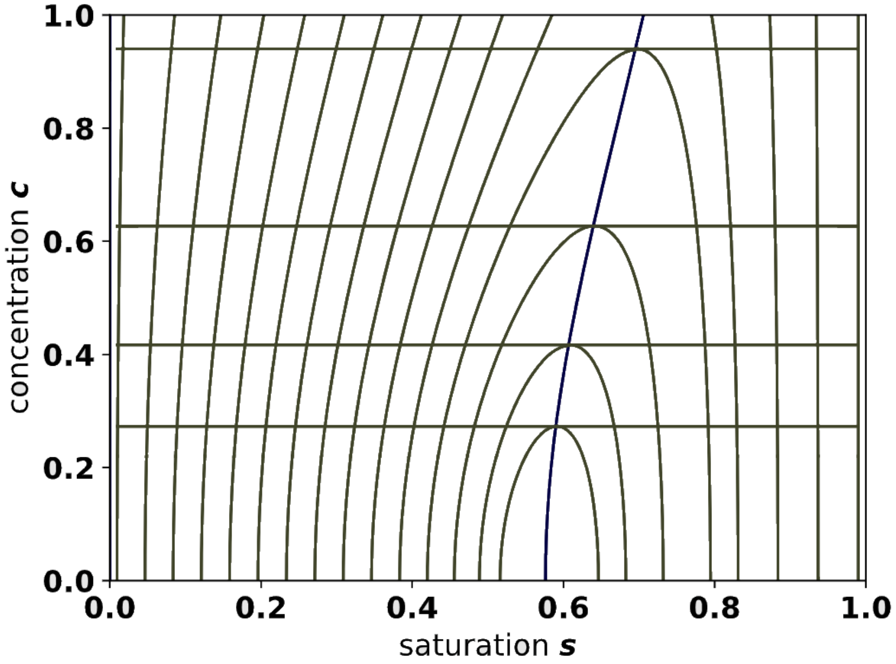
- $\lambda^c = f/s$
- Are contact discontinuities
- Riemann invariant $f/s = const$
- "contact" family




- For both families, the rarefaction and shock curves coincide! But in a different way (Temple'1983)
- Any solution for a Riemann problem is a combination of $s$ and $c$ waves

# Non-strictly hyperbolic system

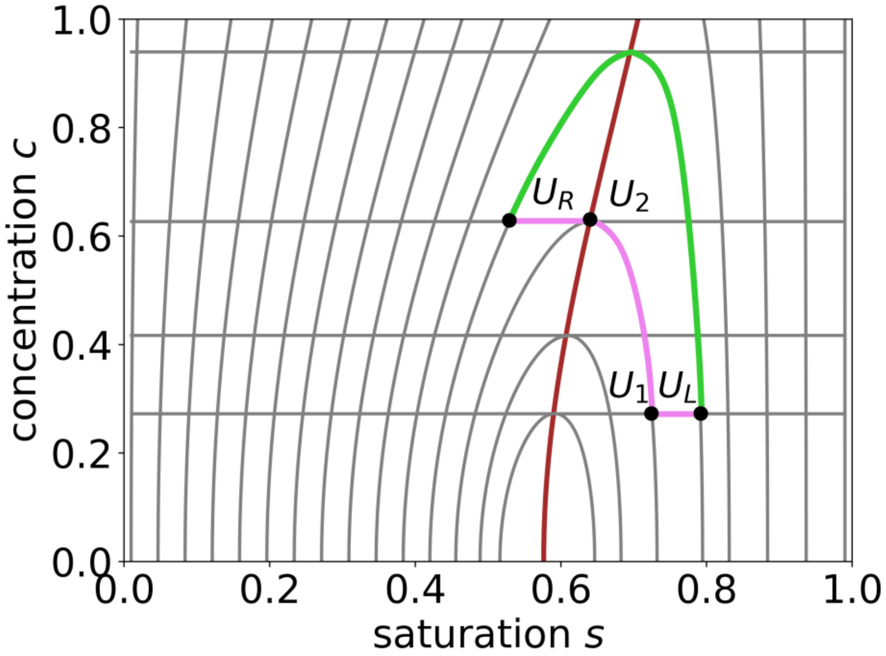


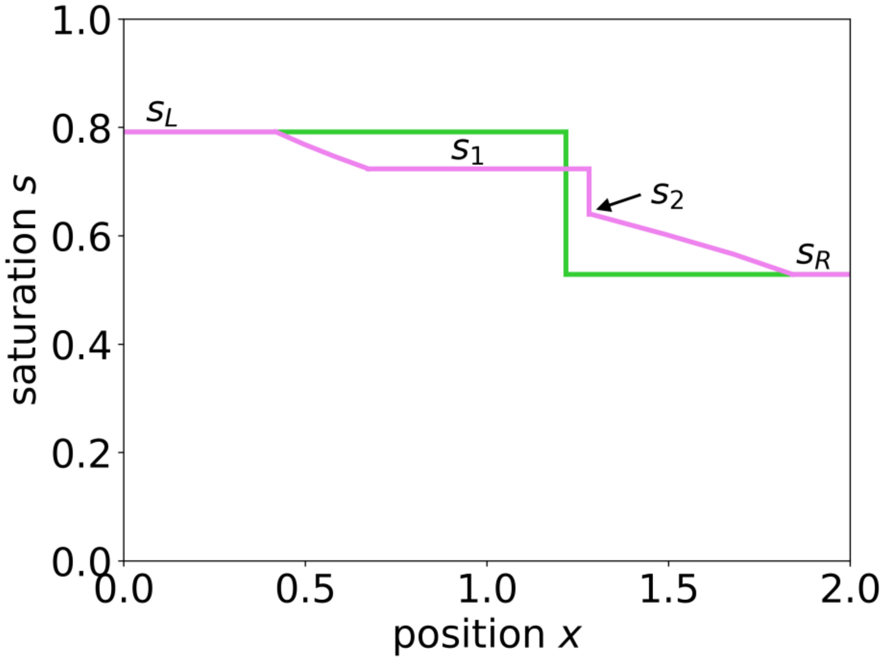




The coordinate system of wave curves is singular and the wave speeds coincide on a co-dimension one curve (coincidence locus): $\lambda^s = f_s' = f/s = \lambda^c$

$s$ and $c$ waves are tangent on coincidence curve

# Non-uniqueness of solutions




A contact discontinuity between $U_-$ and $U_+$ is admissible if and only if:

Criterion 1 (Isaacson): either $U_-, U_+ \in \{\lambda^s \geq \lambda^c\}$ or $U_-, U_+ \in \{\lambda^s \leq \lambda^c\}$

Criterion 2 (de Souza-Marchesin): $c$ is continuous and monotone along the sequence of contact curves, connecting $U_-$ and $U_+$

*What is the (physical) motivation of these criteria?*

# Main result

Polymer model

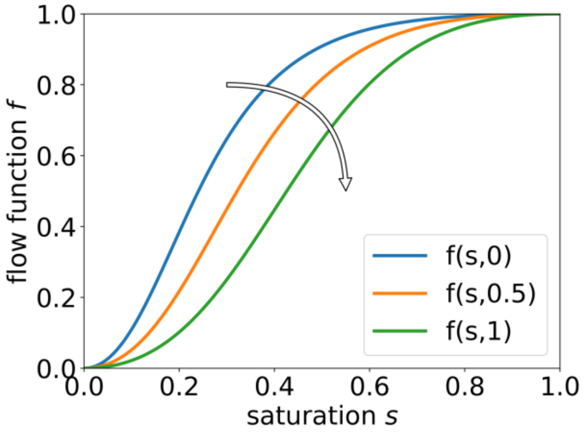
$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned} \qquad (M_\alpha)$$

Criterion 3: vanishing adsorption (Petrova-Marchesin-Plohr):
A contact discontinuity between $U_-$ and $U_+$ for $M_0$ is admissible if and only if it is the $L^1_{loc}$ limit of a family of admissible solutions for a Riemann problem for $M_\alpha$ as $\alpha \to 0$.

**Theorem 1 (P., Marchesin, Plohr, 2022, arxiv:2211.10326)**

If $f$ satisfies the monotonicity assumption $f_c{}' < 0$,
then the set of admissible Riemann solutions for $M_0$ is the same for criteria 1, 2 and 3.

Case 1:




Corollary: any solution to a Riemann problem for $M_0$ exists and is unique.

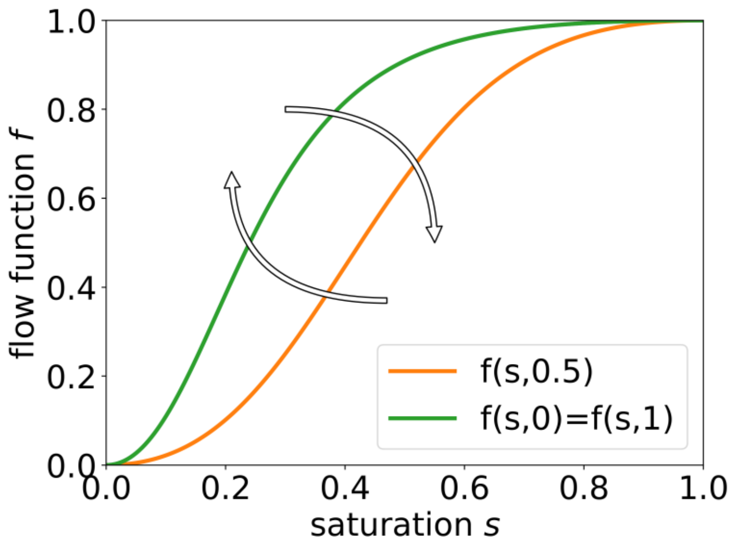
Question: what happens when $f$ is non-monotone in $c$?

# What happens when $f$ is non-monotone in $c$?

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$

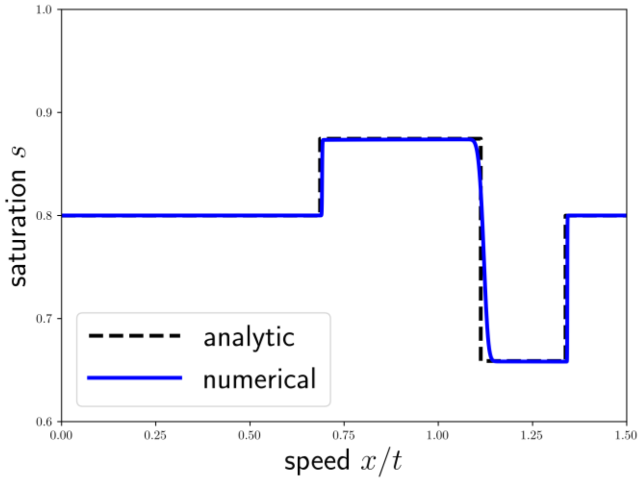
Example: “boomerang”

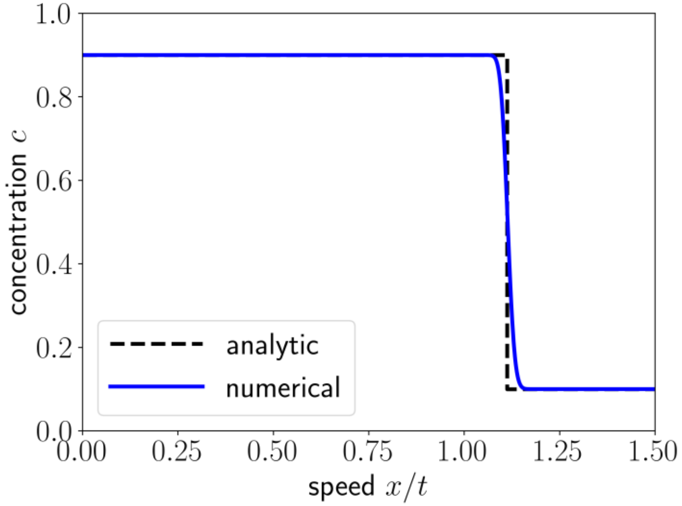
$$f(s,c) = \frac{s^2}{s^2+\mu(c)(1-s)^2} \qquad \text{with} \qquad \mu(c) = 1 + 4c(1-c)$$

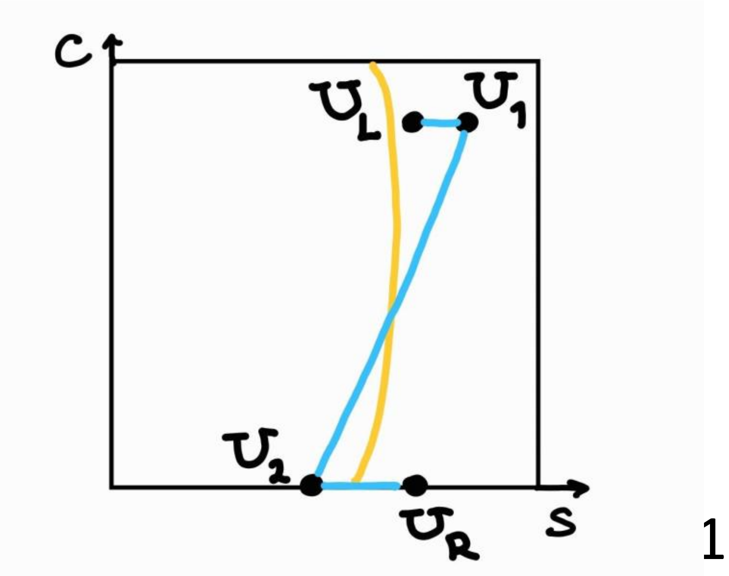



Riemann problem: $s_L = s_R = 0.82,\, c_L = 1,\, c_R = 0$

Results of numerical modelling:

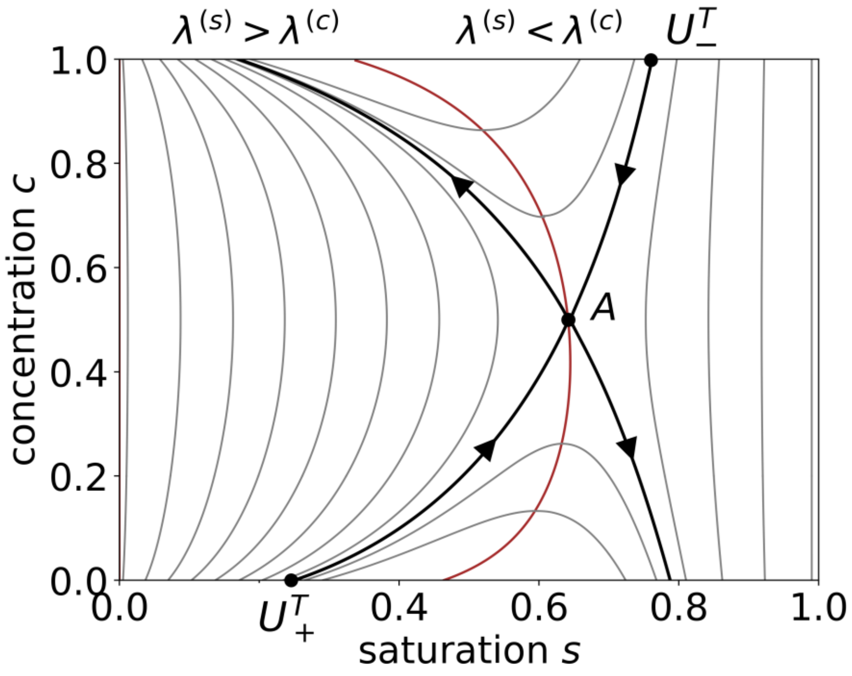


# …a more careful look…

$$\begin{aligned} s_t + \quad f(s,c)_x &= 0 \\ (sc)_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$




Let's call the corresponding contact discontinuity undercompressive.

Is this contact discontinuity admissible by the existing criteria?

Criteria 1: NO
Criteria 2: YES
Criteria 3: YES

**Theorem 2 (P., Marchesin, Plohr, 2022, arxiv:2211.10326)**

The undercompressive contact discontinuities satisfy the vanishing adsorption admissibility criterion

# Main step in proof of Thm 2

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= 0 \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= 0 \end{aligned}$$

Add diffusion terms

$$\begin{aligned} s_t \quad + \quad f(s,c)_x &= \varepsilon_1 s_{xx} \\ (sc + \alpha\, a(c))_t + \big(cf(s,c)\big)_x &= \varepsilon_1 (c s_x)_x + \varepsilon_2 c_{xx} \end{aligned}$$

$$\varepsilon_1, \varepsilon_2 \to 0, \qquad k = \frac{\varepsilon_1}{\varepsilon_2}$$

Travelling wave ansatz

$$s = s(x - \sigma t) = s(\xi), \qquad c = c(x - \sigma t) = c(\xi)$$

$$\begin{cases} \alpha \cdot s_\xi = f - \sigma(s + d_1) \\ c_\xi = \dfrac{\varepsilon_1}{\varepsilon_2} \cdot \sigma \cdot (d_1 c - d_2 - a(c)) \end{cases}$$

**Theorem 3 (Bakharev, Enin, P., Rastegaev, 2023, JHDE)**

For any $\mathrm{k} = \frac{\varepsilon_1}{\varepsilon_2} > 0$ there exists $s_-(k)$ and $s_+(k)$ and velocity $\sigma(k)$ such that there exists a travelling wave, connecting two saddle points $(s_-(k), 1)$ and $(s_+(k), 0)$ with velocity $\sigma(k)$.

# References Muito obrigada pela sua atenção!


**Own works:**

1. F. Bakharev, A. Enin, Y. Petrova, and N. Rastegaev. Impact of dissipation ratio on vanishing viscosity solutions of the Riemann problem for chemical flooding model. Journal of Hyperbolic Differential Equations, 20:1–26, 2023.
2. Y. Petrova, B. Plohr, and Marchesin D. Vanishing adsorption admissibility criterion for contact discontinuities in the polymer model. arXiv:2211.10326.

**Other references:**

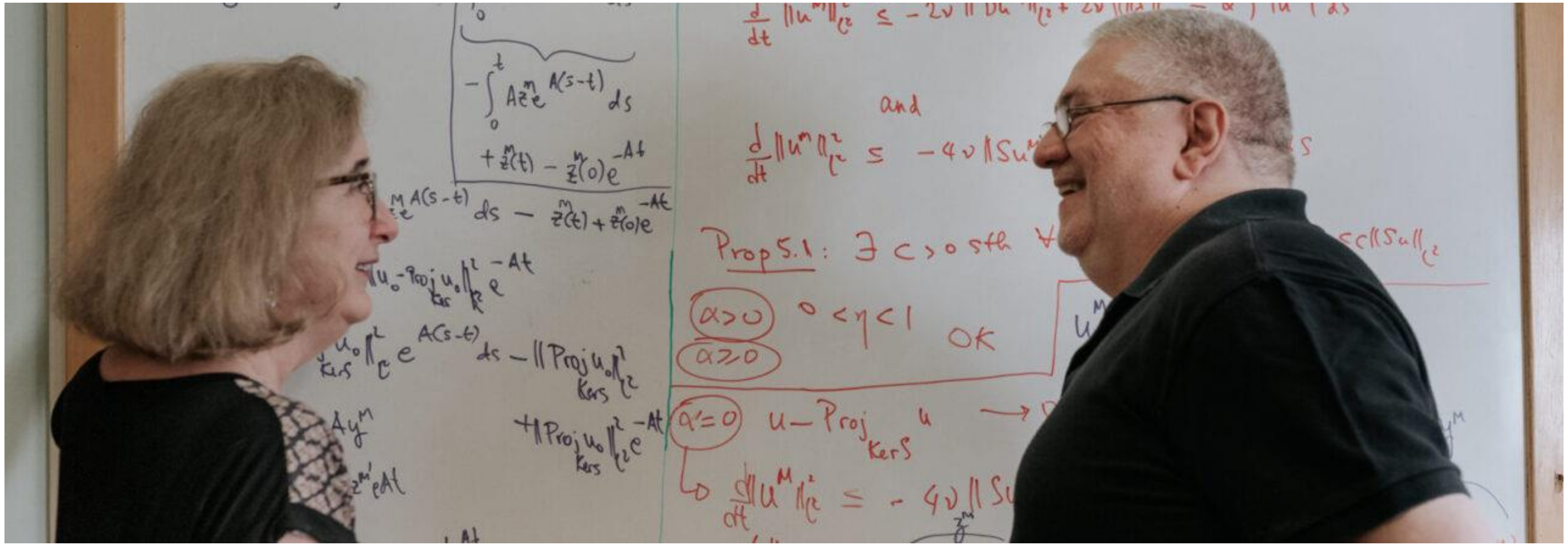
**Polymer model:**

1. W. Shen. On the Cauchy problems for polymer flooding with gravitation. Journal of Differential Equations, 261(1):627–653, 2016.
2. W. Shen. On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. Nonlinear Differential Equations and Applications NoDEA, 24(4):37, 2017.
3. B. Temple. Global solution of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws. Advances in Applied Mathematics, 3(3):335–375, 1982.
4. T. Johansen and R. Winther. The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. SIAM Journal on Mathematical Analysis, 19(3):541–566, 1988.
5. Keyfitz, B.L. and Kranzer, H.C., 1980. A system of non-strictly hyperbolic conservation laws arising in elasticity theory. Archive for Rational Mechanics and Analysis, 72(3), pp.219-241.
6. E. L. Isaacson, Global solution of a Riemann problem for a non-strictly hyperbolic system of conservation laws arising in enhanced oil recovery, Rockefeller University, New York, NY, preprint (1981).

# Feliz aniversario, Helena e Milton!!!

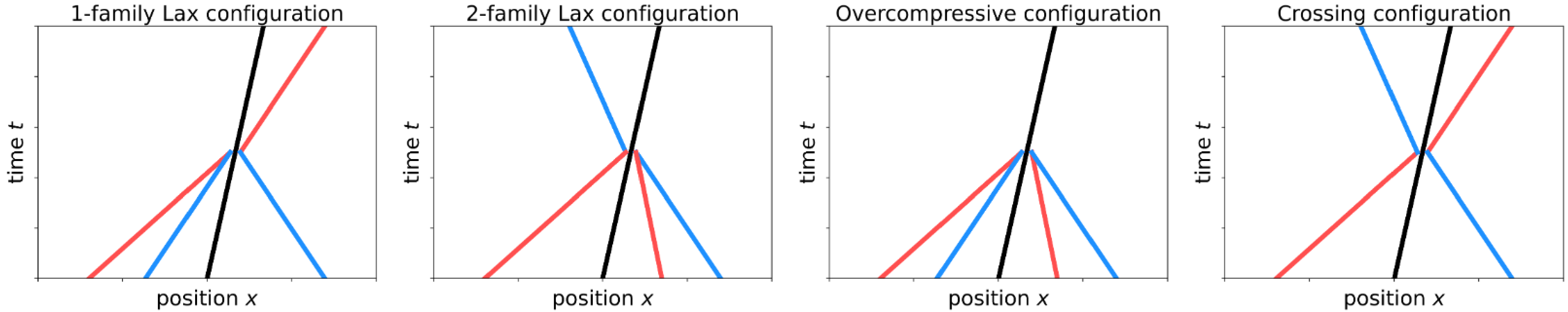
Parabéns pra vocês
Nesta data querida
Muitas felicidades
Muitos anos de vida

É pique, é pique
É pique, é pique, é pique
É hora, é hora
É hora, é hora, é hora
Ra-tim-bum!



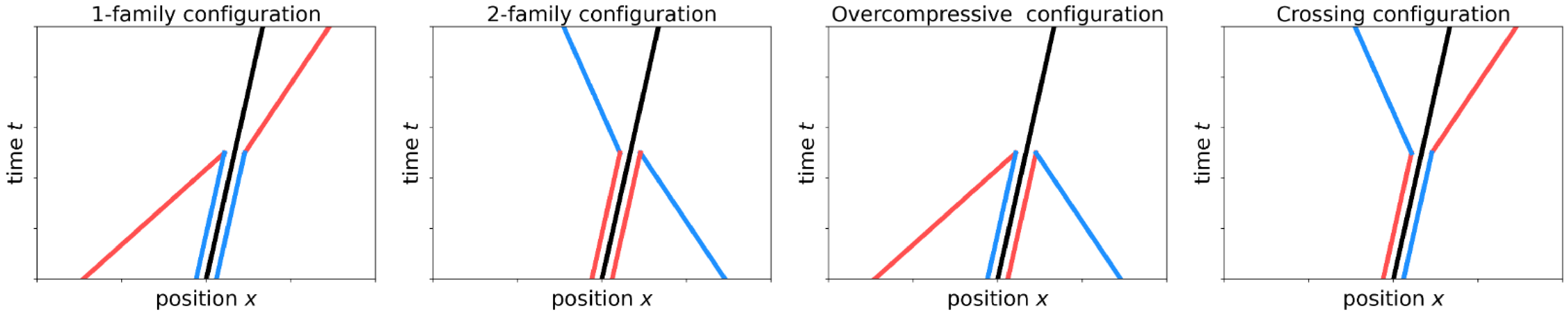

$$H + M = \text{Lopes}^2$$

# Types of shocks




- 1-family Lax: $\lambda_1(U_-) > \sigma > \lambda_1(U_+)$ along with $\sigma < \lambda_2(U_-)$ and $\sigma < \lambda_2(U_+)$
- 2-family Lax: $\lambda_2(U_-) > \sigma > \lambda_2(U_+)$ along with $\sigma > \lambda_1(U_-)$ and $\sigma > \lambda_1(U_+)$
- overcompressive: $\lambda_1(U_-) > \sigma > \lambda_1(U_+)$ and $\lambda_2(U_-) > \sigma > \lambda_2(U_+)$
- crossing: $\lambda_2(U_-) > \sigma > \lambda_1(U_-)$ and $\lambda_1(U_+) < \sigma < \lambda_2(U_+)$.

# Types of contact discontinuities




- 1-family: $\lambda_1(U_-) = \sigma = \lambda_1(U_+)$ along with $\sigma < \lambda_2(U_-)$ and $\sigma < \lambda_2(U_+)$;
- 2-family: $\lambda_2(U_-) = \sigma = \lambda_2(U_+)$ along with $\sigma > \lambda_1(U_-)$ and $\sigma > \lambda_1(U_+)$;
- overcompressive: $\lambda_1(U_-) = \sigma > \lambda_1(U_+)$ and $\lambda_2(U_-) = \sigma > \lambda_2(U_+)$;
- crossing: $\lambda_2(U_-) > \sigma = \lambda_1(U_-)$ and $\lambda_1(U_+) = \sigma < \lambda_2(U_+)$.