Problem solving session №4, 20 May 2023.

In this session let's concentrate on the applications of the maximum principle for linear parabolic equations:

$$\partial_t u = \Delta u + b \cdot \nabla u + cu, \qquad x \in \Omega \subset \mathbb{R}^N, t > 0. \tag{1}$$

Here $b = b(t,x)$ and $c = c(t,x)$ are continuous bounded functions. The domain Ω is either a bounded connected open set or $\mathbb{R}^N$. Using the maximum principle, we obtained the comparison principle for the semilinear parabolic equations, e.g. reaction-diffusion equations ($f \in C^1$ in u):

$$\partial_t u = \Delta u + f(t,x,u). \tag{2}$$

Theorem 1 (Weak maximum principle)**.** *Let u be a subsolution of (1). If $u(0,x) \le 0$, then $u(t,x) \le 0$ for $t > 0$.*

Theorem 2 (Weak comparison principle)**.** *Let u be a subsolution of (2) and v be a supersolution of (2). If $u(0,x) \le v(0,x)$, then $u(t,x) \le v(t,x)$ for $t > 0$.*

Here are some problems to solve using these theorems:

1. (Uniqueness for semilinear problems)
   Let $\Omega \subset \mathbb{R}^N$ be bounded, $f \in C^1(\mathbb{R})$, $u_0 \in C^0(\bar{\Omega})$. Prove that the problem

$$\begin{cases} \partial_t u = -\Delta u + f(u), & \text{in } D = \Omega \times (0,T], \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } \partial\Omega \times (0,T], \end{cases}$$

   has at most one solution $u \in C^2(D) \cap C^1(\bar{D})$.

2. (Upper bound on solution for linear problems)
   Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $u(t,x)$ be the solution of the initial boundary value problem

$$\begin{cases} u_t = \Delta u + b \cdot \nabla u + c(x)u, & \text{in } \Omega \times (0,+\infty), \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } x \in \partial\Omega \times (0,+\infty). \end{cases}$$

   Assume that the function $c(x)$ is bounded, with $c(x) \le M$ for all $x \in \Omega$. Prove that $u(t,x)$ satisfies

$$|u(t,x)| \le \|u_0\|_{L_\infty} e^{Mt}, \qquad \text{for all } t > 0 \text{ and } x \in \Omega.$$

3. (Global solution vs. blow-up for reaction-diffusion equations)
   Let u be a solution to the following reaction-diffusion equation

$$\begin{cases} \partial_t u = \Delta u + u^2, & \text{in } D_T = \Omega \times (0,T], \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \times (0,T]. \end{cases}$$

   Does the solution u blow-up in finite time?

4. (Asymptotics for the heat equation)
   Let $\Omega = B_1(0) \subset \mathbb{R}^N$ and suppose $u \in C^2(\Omega \times (0,+\infty)) \cap C^0(\bar{\Omega} \times [0,+\infty))$ satisfies for some $M > 0$:

$$\begin{cases} \partial_t u = \Delta u, & \text{in } \Omega \times (0,+\infty), \\ |u| \le M, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } x \in \partial\Omega \times (0,+\infty). \end{cases}$$

   Prove that $u(x,t) \to 0$ as $t \to \infty$ uniformly in x.

   Hint: combine the functions $2 - |x|^2$ and e^{nt} and construct a supersolution to the heat equation with appropriate behavior at $+\infty$.