Problem solving session №3, 19 May 2023.

In this session let's concentrate on the applications of the maximum principle for ODEs with second order differential operators of the form:

$$L = -\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x), \qquad x \in (a,b) \subset \mathbb{R}.$$

Here $g(x)$ and $h(x)$ are bounded functions.

Theorem 1 (maximum principle)**.** *Let $h \equiv 0$ and $Lu \le 0$. Then if there exists $c \in (a,b)$ such that $u(c) = \max u(x)$ for $x \in [a,b]$, then $u \equiv \max u(x)$.*

1. Does the differential operator L defined on the interval $[a,b] \subset \mathbb{R}$ provide a maximum principle? That is: if for $u \in C^2[a,b] \cap C^0(a,b)$ we have $Lu \le 0$, then maximum of u on $[a,b]$ is obtained on the boundary (either at $x = a$ or at $x = b$).

   a) $L = -\dfrac{d^2}{dx^2} - 1;$ b) $L = -\dfrac{d^2}{dx^2} + 1.$

An important application of (different forms of) maximum principle is the following comparison principle for sub and supersolutions.

Theorem 2. *Let $f \in C^1((a,b) \times \mathbb{R})$, and let u be a subsolution and v be a supersolution, that is:*

$$Lu \le f(x,u); \qquad\qquad Lv \ge f(x,u).$$

Then if $u(x) \le v(x)$ for all $x \in [a,b]$, and there exists $c \in (a,b)$ such that $u(c) = v(c)$, then $u \equiv v$.

In other words, a sub-solution and a super-solution can not touch at a point: either $u \equiv v$ or $u < v$. This "untouchability" of a sub-solution and a super-solution is very helpful. See problems 2 and 3 (and the so-called sliding method).

2. Consider the boundary-value problem:

$$\begin{cases} -u'' = e^u, & 0 < x < L, \\ u(0) = u(L) = 0. \end{cases} \tag{1}$$

   Prove that if L is sufficiently big, there does not exist a non-negative solution to this boundary-value problem. Proceed by the following steps:

   (a) Write a problem in terms of function $w = u + \varepsilon$:

$$\begin{cases} -w'' = e^{-\varepsilon} e^w, & 0 < x < L, \\ w(0) = \varepsilon, \quad w(L) = \varepsilon. \end{cases} \tag{2}$$

   (b) Show that functions $v_\lambda(x) = \lambda \sin(\pi x / L)$ satisfy

$$\begin{cases} -v_\lambda'' = \frac{\pi^2}{L^2} v_\lambda, & 0 < x < L, \\ v_\lambda(0) = 0, \quad v_\lambda(L) = 0. \end{cases} \tag{3}$$

   (c) Show that for big enough $L > 0$ and small enough $\lambda > 0$ the solution w of the problem (2) is a supersolution of the problem (3).

   (d) *(Sliding method)* Start increasing $\lambda > 0$ and consider the first value λ_0 such that the graphs of w and v_λ touch each other. Come to a contradiction.

   (e*) Show that there exists $L_1 > 0$ so that non-negative solution of problem (1) exists for all $0 < L < L_1$ and does not exist for all $L > L_1$.

3. Using sliding method from the previous exercise, prove that the solution u of the boundary value problem:

$$\begin{cases} -u'' - cu' = f(u), & -L < x < L, \\ u(-L) = 1, \quad u(L) = 0. \end{cases}$$

   is unique.

Read more material about different kinds of maximum principle on the web-page on Miles Wheeler — Course "Theory of Partial Differential Equations"