Exercise session №2, 5 May 2023.

In this session let us concentrate on the systems of conservation laws ($U \in \mathbb{R}^m$, $m > 1$, $F : \mathbb{R}^m \to \mathbb{R}^m$):

$$U_t + F(U)_x = 0, \tag{1}$$

with Riemann initial data ($U_l, U_r \in \mathbb{R}^m$ — fixed):

$$U(x,0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases} \tag{2}$$

1. Consider a linear wave equation $w_{tt} - c^2 w_{xx} = 0$.

   It can be rewritten in the form (1) for $U = \begin{pmatrix} w_x & w_t \end{pmatrix}^T$ as follows:

   $$U_t + AU_x = 0, \qquad U = \begin{pmatrix} v \\ u \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

   (a) Find eigenvalues and eigenvectors of A;

   (b) Show that for $c \neq 0$ the system is strictly hyperbolic;

   (c) Show that the system is linearly degenerate;

   (d) Find explicit solution to (global) Riemann problem (2) for any $U_l, U_r \in \mathbb{R}^m$.

2. Consider a nonlinear wave equation $w_{tt} - (p(w_x))_x = 0$ with $p' < 0$, $p'' > 0$.

   This model comes from gas dynamics, where p is the pressure and typically $p(w) = w^{-\gamma}$ for $\gamma \geq 1$.

   It can be rewritten in the form (1) for $U = \begin{pmatrix} w_x & w_t \end{pmatrix}^T$ as follows:

   $$U_t + F(U)_x = 0, \qquad U = \begin{pmatrix} v \\ u \end{pmatrix}, \qquad F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}, \qquad DF(U) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$

   (a) Find eigenvalues and eigenvectors of $DF(U)$;

   (b) Show that if $p' \neq 0$ the system is strictly hyperbolic;

   (c) Show that if $p'' \neq 0$ the system is genuinely nonlinear for each characteristic family;

   (d) For fixed U_l find explicitly shock curves. Which part of them correspond to admissible shock waves (according to Lax admissibility criterion)? Draw 1-shock and 2-shock curves in (v, u)-plane. Draw 1-shock and 2-shock waves in (x, t)-plane.

   (e) For fixed U_l find explicitly rarefaction curves. Which part of them correspond to rarefaction waves? Draw 1-rarefaction and 2-rarefaction curves in (v, u)-plane. Draw 1-rarefaction and 2-rarefaction waves in (x, t)-plane.

   (f) Show that shock and rarefaction curves from items (d) and (e) divide the neighbourhood of U_l into 4 regions. Draw the solution to a (local) Riemann problem in (x, t)-plane considering U_r lies in one of these 4 regions.

   (g*) Show that if

   $$\int_{v_l}^{\infty} \sqrt{-p'(y)}\, dy = \infty,$$

   then there exists a solution to a global Riemann problem, that is for any U_l and U_r (not necessarily sufficiently close to each other). Is it unique?