Exercise session №1, 4 April 2023.

In this session let us concentrate on the Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{1}$$

with different initial (or boundary) conditions.

Definition 1. *A shock-wave solution, connecting states u_L and u_R and moving with speed c, is the solution of the form:*

$$u(x,t) = \begin{cases} u_L, & x < ct, \\ u_R, & x > ct, \end{cases}$$

For a general single conservation law $u_t + (f(u))_x = 0$ there is a relation between u_L, u_R and c:

$$\text{(Rankine-Hugoniot condition = RH)} \qquad c = \frac{f(u_L) - f(u_R)}{u_L - u_R}. \tag{2}$$

1. Construct a shock-wave solution to the Burgers equation with the following conditions

$$u(x,t) = \begin{cases} 1, & x = 0, \\ 0, & t = 0, \end{cases}$$

2. Consider the Burgers equation with the following initial conditions:

$$\text{(Riemann problem)} \qquad u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

   Construct:

   a) a smooth self-similar solution of the form: $u = v\left(\frac{x}{t}\right)$;
   b) a shock-wave solution.

   So we have at least two solutions! Which one is "correct"?

3. Construct infinitely-many solutions to the following initial-value problem:

$$\text{(Riemann problem)} \qquad u(x,0) = \begin{cases} -1, & x < 0, \\ +1, & x > 0. \end{cases}$$

   Remark 1. *A natural question to ask is what EXTRA condition do we need to choose one solution? Such condition is usually called an "entropy" condition. An example of such condition is as follows: there exists a constant $E \in \mathbb{R}$ (independent of x, t and a):*

$$\frac{u(x+a,t) - u(x,t)}{a} \le \frac{E}{t}, \qquad a > 0, \quad t > 0. \tag{3}$$

   This condition implies that if we fix $t > 0$ and let x go from $-\infty$ to $+\infty$, then we can only jump down. Let us call the solutions that satisfy condition (3) the "entropy" solutions.

4. Which of the solutions from exercises 1–3 are entropy solutions?

5. Construct an entropy solution to the Burgers equation with the following initial conditions

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \in [0,1], \\ 0, & x > 1, \end{cases}$$

   Consider two cases: $t \in [0,2]$ and $t \ge 2$.

6. (Irreversibility) Let the solution at $t = 1$ be equal to:

$$u(x,1) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{4}$$

   Construct infinitely-many different initial conditions $u(x,0)$ (and draw them up to time $t = 1$) such that at $t = 1$ the solution coincides with (4).