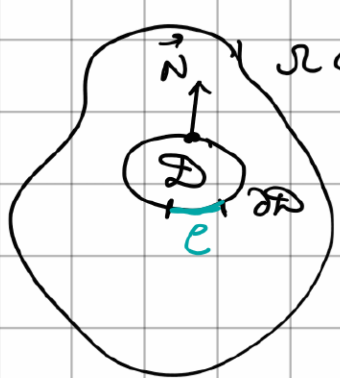
Lecture 5: Conservation & Balance laws

Plan:
1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

(1) Balance law



$\mathcal{D} \subset \Omega$ with Lipshitz boundary (smooth)

$\vec{N}$ - normal vector towards the exterior of the domain $\mathcal{D}$

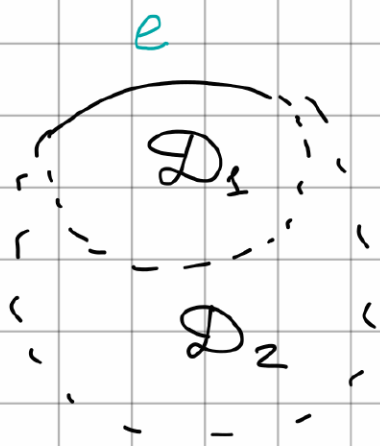
$$\boxed{\text{production in } \mathcal{D} = \text{flux through the boundary of } \partial\mathcal{D}}$$

- production in $\mathcal{D}$ is some measure (Radon) $\mathcal{P}$
- flux

$$F_{\mathcal{D}}(e) = \int_{e} q_{\mathcal{D}}(x)\, dS(x)$$

$$\mathcal{P}(\mathcal{D}) = \int_{\partial\mathcal{D}} \underline{q_{\mathcal{D}}(x)}\, dS(x) \qquad (*)$$

Assume:



$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$

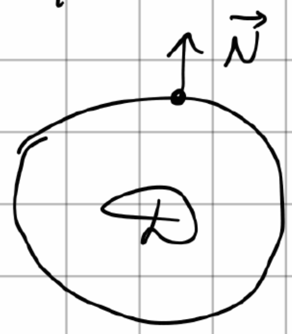
$\forall x \in e$

Take-home (Tuesday)

$$\boxed{\operatorname{div} A = 0}$$

Miracles :

Consequences of (*) :



(1) $\exists\ a_{\vec{N}}(x) = q_{\mathcal{D}}(x)$

$\forall x \in \Omega$

for any $\mathcal{D} \subset \Omega$ s.t.

$\mathcal{D}$ has $\vec{N}$ as a normal vector at x.

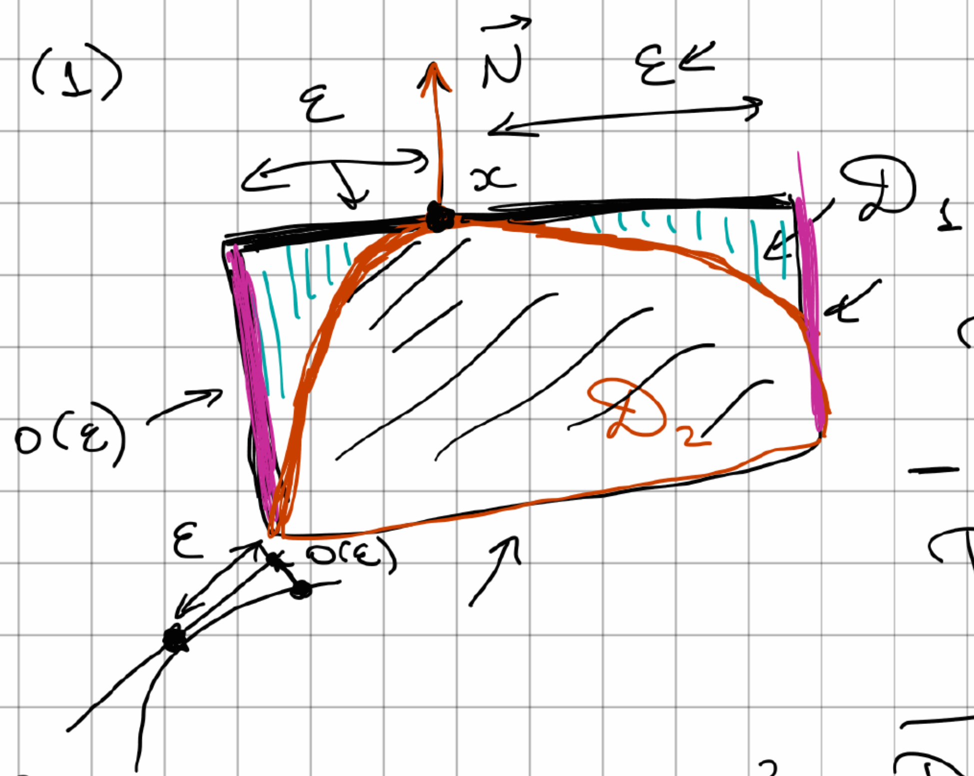
(2) $\exists\ \vec{A}(x) : \Omega \to \mathbb{R}^d$:

$a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$

(3) $\exists$ PDE: $\operatorname{div} \vec{A} = P$

$$P(\mathcal{D}) = \int_{\partial\mathcal{D}} \underbrace{q_{\mathcal{D}}(x)}_{\substack{\parallel \\ \vec{A}(x)\cdot\vec{N}}} dS(x)$$

(1)



$\varepsilon \to 0$

$q_{\mathcal{D}_1}(x) \overset{?}{=} q_{\mathcal{D}_2}(x)$

(*)

$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$

$-$

$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$

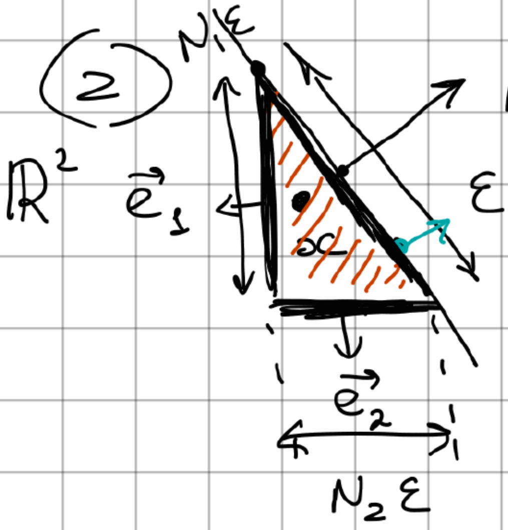
$\int \sim o(\varepsilon)$

$\varepsilon^2 \sim P(\ ) = \int q_{\mathcal{D}_1}(x) dS - \int q_{\mathcal{D}_2}(x) dS$

$\Rightarrow \int q_{\mathcal{D}_1}(x) dS(x) = \int q_{\mathcal{D}_2}(x) dS(x)$

$$\Rightarrow \quad \exists \; a_{\vec N}(x) = q_{\mathcal{D}}(x)$$ <u>Cauchy</u>

(2)



$$\vec N = (N_1, N_2)$$

$$\varepsilon^2 \sim \mathcal{P}(\triangle) = \int_{\triangle} a_{\vec N}(x)\,dx$$

$$a_{\vec N}(x)\cdot \underline{\cancel{\varepsilon}} = a_{e_2}(x)\cdot \underline{N_2 \cancel{\varepsilon}} + a_{e_1}(x)\cdot N_1 \cancel{\varepsilon}$$

$$\Rightarrow \quad a_{\vec N}(x) = a_{e_1}(x)\,N_1 + a_{e_2}(x)\,N_2$$

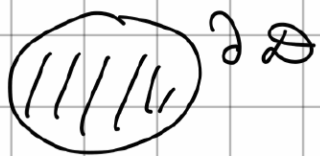
$$\Rightarrow \quad a_N(x) = \vec A(x)\cdot \underline{\vec N}$$

$$A(x) = (a_{e_1}, a_{e_2})$$

(3)

$$\int_{\partial\mathcal{D}} \vec A(x)\cdot \vec N \, dS(x) = \int_{\mathcal{D}} \mathrm{div}(\vec A)\,dx$$

$\uparrow$ Green-Gauss theorem



$$\| \quad \mathcal{P} = \int_{\mathcal{D}} p(x)\,dx$$

$$\Rightarrow \quad \mathrm{div}(\vec A) = p$$ – balance law

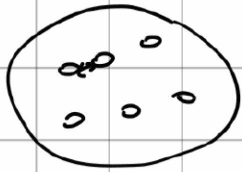
$$\boxed{\mathrm{div}(\vec A) = 0}$$ – conservation law

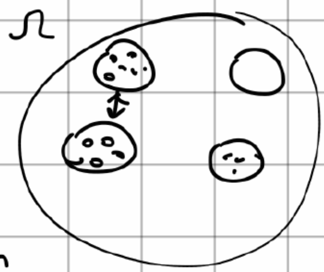
Dafermos

<u>Example 1</u> : Fluid flow, continuum mechanics

different scales

1. atoms / molecules
2. representative small volume
3. domain (macroscale)





- <u>Eulerian</u> vs. Lagrangian point of view

<u>Eulerian</u> : (x,t) - fix

- velocity : $u(x,t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$
  has units $\left[\frac{L}{T}\right]$
- density : $\rho(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[\frac{M}{L^d}\right]$
- pressure : $p(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[M\, L^{-d+2}\, T^{-2}\right]$

<u>Lagrangian</u> : particles, $a \in \mathbb{R}^d$
trajectories of particles

flow map $X(t,a) = (X_1, \dots, X_d)$ - position of particle a at time t

$$(**) \begin{cases} \partial_t X(t,a) = u(t, X(t,a)) & \leftarrow \text{ODE} \\ X(0,a) = a \end{cases}$$

ODE theory (Cauchy-Lipschitz theorem) :
$u \in C_t \mathrm{Lip}_x \Rightarrow \exists !$ solution to $(**)$

$X(t,\cdot)$ – is C^1-diffeo : $\mathbb{R}^d \to \mathbb{R}^d$

Define inverse: $A(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$$A(t, X(t,a)) = a \qquad X(t, A(t,x)) = x$$

$$\forall x, a \in \mathbb{R}^d$$

"back-to-labels" map (a – "labels")

Incompressibility condition : " $\operatorname{div} u = 0$ "

Take $V \subset \Omega$ – volume of fluid

$$V(t) = X(t, V) = \{ X(t,a) : a \in V \}$$

Def: velocity field is called incompressible if

$$\rightarrow \quad |V(t)| \equiv |V|$$

$\uparrow$ Lebesgue measure of V

Lemma: $u \in C_t \operatorname{Lip}_x$

u is incompressible $\iff$ $\operatorname{div} u = 0$ (u is divergence-free)

Proof:

$\Gamma \blacktriangleright$

$$V(t) = \int_{V(t)} 1 \cdot dx \quad ; \quad a \in V \subset \mathbb{R}^d$$

$$\int_{V(t)} f(x,t)\, dx = \int_V f(X(t,a),t) \cdot \boxed{\det\left(\nabla_a X\right)(t,a)}\, da$$

$a \mapsto X(t,a)$ $\qquad$ $\underset{J(t,a)}{\shortparallel}$

$$\rightarrow \quad J(t,a) = \sum_{i_1,\dots i_d = 1}^{d} \underline{\varepsilon_{i_1 \dots i_d}} \frac{\partial X_{i_1}}{\partial a_1} \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}$$

Exercise : $\partial_t J(t,a) = J(t,a) \cdot (\operatorname{div} a)(t, X(t,a))$

Corollary: $J(t,a) \equiv 1 \iff \underline{\mathrm{div}(u) = 0}$

$$J(t,a) = J(0,a) \cdot e^{\int_0^t (\mathrm{div}\, a)(s, X(s,a))\,ds}$$

$$J(0,a) = 1$$

$$J(t,a) = J(0,a) \quad \forall t \Rightarrow \mathrm{div}(a) = 0$$

$$V(t) = \int_{V(t)} 1\,dx = \int_V J(t,a)\,da = \int_V da = V$$

iff $\mathrm{div}(a) = 0$ ∎

Transport equation

Let's $f(t,x): [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ - scalar

Eulerian: $\partial_t f$ - change of f at (t,x)

Lagrangian: $\partial_t f(t, X(t,a)) =: D_t f$ - convective derivative

$$= \partial_t f + u \cdot \nabla f$$

$$V \xrightarrow{t} V(t)$$

Thm (transport thm):

u - velocity field, $u \in C^1$; f be C^1

$V(t)$ is pushforward of V by the flow map $X(t,a)$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_{V(t)} (\partial_t f + \mathrm{div}(fu))(t,x)\,dx$$

Proof:

$$\int_{V(t)} f(x,t)\,dx = \int_V f(X(t,a),t)\,\underline{J(t,a)}\,da$$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_V \underline{\left(D_t f\right)}\,(X(t,a),t)\, J(t,a)$$

$$+ \int_V f(X(t,a),t)\cdot \underbrace{\partial_t J(t,a)}_{J(t,a)\cdot \operatorname{div}(u)}\,da =$$

$$= \int_V \Big(\partial_t f + \underbrace{u\cdot\nabla f + f\cdot \operatorname{div}(u)}_{\operatorname{div}(uf)}\Big)(X(t,a),t)\cdot J(t,a)\,da$$

$$= \int_V \Big(\partial_t f + \operatorname{div}(fu)\Big)(X(t,a),t)\cdot J(t,a)\,da =$$

$$= \int_{V(t)} \Big(\partial_t f + \operatorname{div}(fu)\Big)\,dx \qquad \blacksquare$$

Conservation of mass: $\rho(x,t)$

$$m(t,V) = \int_V \rho(x,t)\,dx$$

$$\frac{d}{dt}\, m(t,V(t)) = 0$$

<u>Thm</u>: conservation of mass is equivalent to the following integral eq:

$$\int_{V(t)} \big(\rho_t + \operatorname{div}(\rho u)\big)\,dx = 0$$

If ρ_t and $\operatorname{div}(\rho u)$ are C, then

$$\boxed{\rho_t + \operatorname{div}(\rho u) = 0}$$

<u>Rmk</u>: scalar transport eq

<u>Proof</u>:

$$0 = \frac{d}{dt} m(t, X(t,a)) = \frac{d}{dt}\int_{V(t)} \rho(x,t)\,dx$$

$$= \int_{V(t)} \underbrace{\left(\rho_t + \operatorname{div}(\rho u)\right)}_{\text{are continuous}} dx .$$

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\,dy$$

$$\Rightarrow \quad \rho_t + \operatorname{div}(\rho u) = 0. \qquad \blacksquare$$

<u>Rmk</u>:

$$0 = \rho_t + \operatorname{div}(\rho u) = \rho_t + u\cdot\nabla\rho + \operatorname{div}(u)\rho$$

incompressibility $\Rightarrow$ $\operatorname{div}(u) = 0$

$$D_t \rho = \rho_t + u\cdot\nabla\rho = 0$$

$$\frac{d}{dt}\left(\rho(t, X(t,a))\right) = 0$$

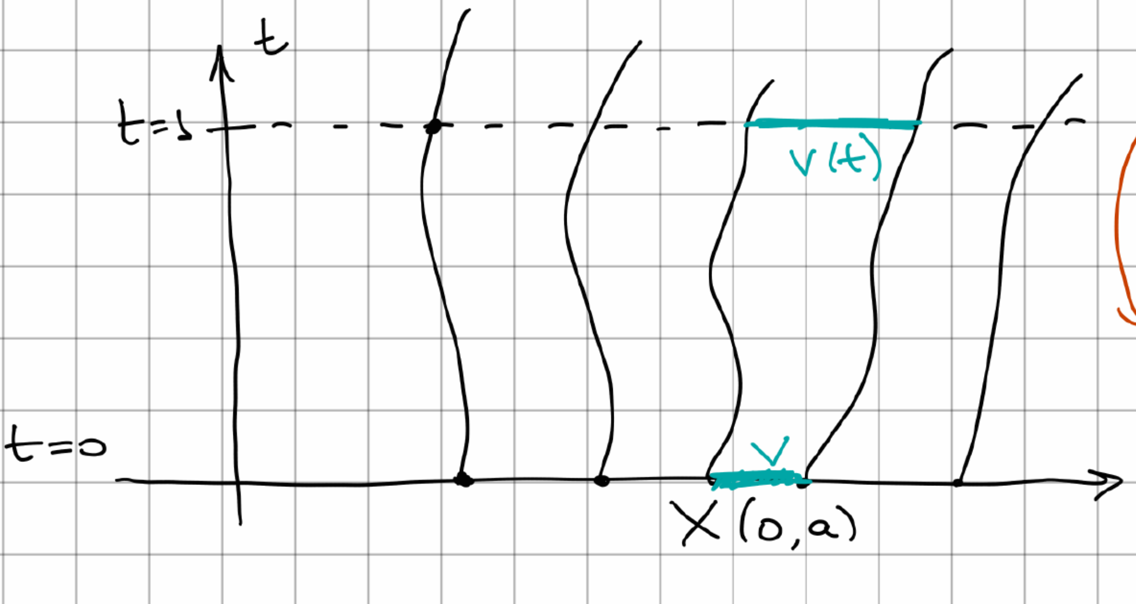
$$\Rightarrow \rho(t, X(t,a)) = \text{const}.$$

Next time

$\rho_t + (\rho \underline{u})_x = 0$

$u = u(\rho)$

$u(\rho) = \frac{\rho}{2} \Rightarrow$ Burgers



$u \in C_t \underline{Lip_x}$

linear in ρ

$\rho_t + \underline{u}\cdot\nabla\rho = 0$

Lecture 6 Last time: Balance laws: $\operatorname{div} A = P$

Conservation laws: $\operatorname{div} A = 0$

Example 1: fluid flow: $a \to X(t,a)$ – flow map under velocity field, $a \in \mathbb{R}^d$

$u(x,t)$ at x

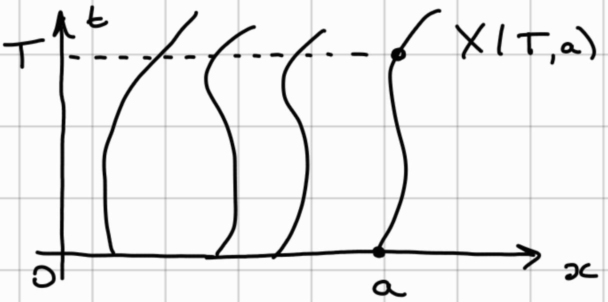
$u(t,x)$ – velocity field

$\rho(t,x)$ – density

$$\begin{cases} \partial_t X = u(X,t) \\ X(0,a) = a \end{cases}$$

• Conservation of mass = scalar transport equation

$$\boxed{\partial_t \rho + \operatorname{div}(\rho u) = 0}$$

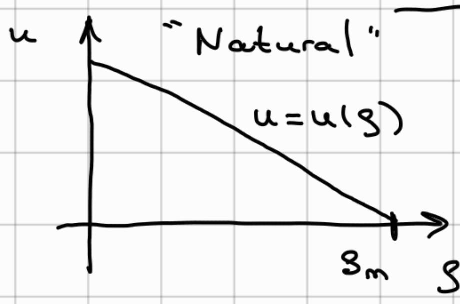


trajectories do not intersect for given $u \in C^0_t \mathrm{Lip}_x$

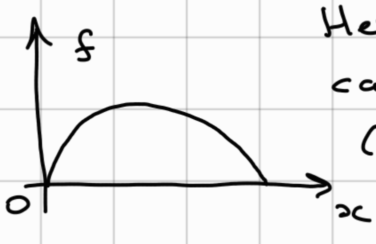
Rmk: $\begin{cases} \operatorname{div} u = 0 \\ \partial_t \rho + \operatorname{div}(\rho u) = 0 \end{cases}$

$\Rightarrow$ $\rho(t, X(t,a)) = \text{const}$
density is conserved along the trajectory for incompressible flow

Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby

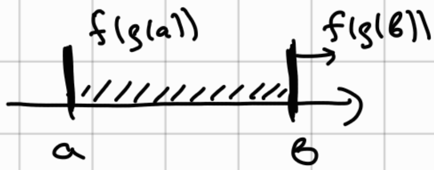


$\Rightarrow$ $\boxed{\rho_t + f(\rho)_x = 0}$ scalar conservation law



Here $f(\rho)$ is called flux (flow function)

ρ_m – density of cars corresp. to "bumper-to-bumber"



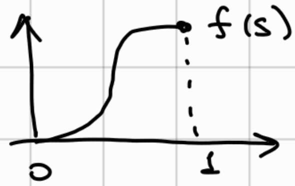
$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = f(\rho(b)) - f(\rho(a))$$

Rmk: 1) taking $u(\rho) = \frac{\rho}{2} \Rightarrow$ Burgers eq: $\rho_t + \left(\frac{\rho^2}{2}\right)_x = 0$

We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water–oil is again

$s_t + (f(s))_x = 0$ for



$f(s)$: $f(0)=0$, $f(1)=1$, $f \uparrow$ and S-shaped

s - water saturation

$f(s)$ - fractional flow function

- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - V(\rho) \sim -\rho_x$$

The simplest law: $u = v(\rho) - \varepsilon \rho_x$, $0 < \varepsilon \ll 1$

which leads to the "weakly" parabolic eq:

$$\rho_t + f(\rho)_x = \varepsilon (\rho \rho_x)_x$$

Example 3: wave equation! $u_{tt} - c^2 u_{xx} = 0$

$$\operatorname{div}(u_t, -c^2 u_x) = 0$$

Consider $U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + A U_x = 0$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Indeed, this is just: $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of A: $\det \begin{vmatrix} 0-\lambda & -1 \\ -c^2 & -\lambda \end{vmatrix} = \lambda^2 - c^2$, $\lambda_{\pm} = \pm c$

They correspond to propagation modes:

This is general fact that we will see in the future:

$U \in \mathbb{R}^d$, $F : \mathbb{R}^d \to \mathbb{R}^d$ $\quad U_t + (F(U))_x = 0$ - system of conservation laws

Then for "smooth" solutions we have:

$$U_t + \underbrace{F'(U)}_{} \cdot U_x = 0$$

eigenvalues of this matrix play an important role!

If they are real, they correspond to velocity of propagation of waves.

Example 4: isentropic (= constant entropy) gas dynamics (p-system)

in Lagrangian coordinates:
$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$$

Rmk: $v_t = u_x \Rightarrow$ (in a simply connected regions) $\exists\, \varphi$: $v = \varphi_x$, $u = \varphi_t$

$$\Rightarrow \quad \varphi_{tt} + (p(\varphi_x))_x = 0$$

$$\varphi_{tt} + p'(\varphi_x)\cdot \varphi_{xx} = 0$$ – nonlinear wave equation

And many other examples:

- conservation of mass, conservation of momentum $\Big\} \Rightarrow \begin{cases} \partial_t u + u\cdot\nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$

  This is Euler equations for ideal fluid (1755, second PDE ?)

- Navier–Stokes eqs (1845): adds viscosity
$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u = \nabla p$$

- gas dynamics
- electromagnetism (Maxwell eqs)
- magneto-hydrodynamics (M.H.D.) – motion of fluid in the presence of electromagnetic field (think of a Sun)

Etc ......

Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad\qquad u_t + u\cdot u_x = 0$$

Observation 1: if $u \in C^1$ for all $t>0$, then u is monotonically nondecreasing in x for all $t>0$.

$$u(x(s), t(s)) = \text{const}$$

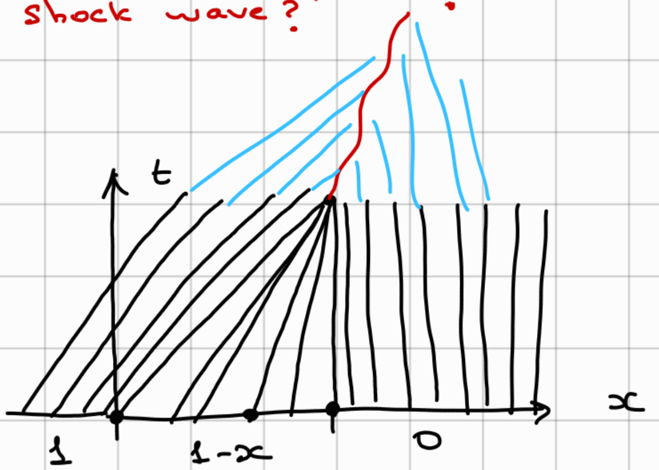
$$u_t \cdot t_s + u_x \cdot x_s = 0$$

$$\begin{cases} t_s = 1 \\ x_s = u \end{cases} \Rightarrow \begin{array}{l} u = \text{const on} \\ \text{straigh lines} \\ x = x_0 + u_0(x_0)\, t \end{array}$$

If $u \in C^1$ for $\forall\, t>0$, then characteristics should not intersect $\Rightarrow$ $u_0(x_1) < u_0(x_2)$ if $x_1 < x_2$ $\Rightarrow$ u_0 is non-decreasing $(u(x,0))$ $\Rightarrow$ $u(x,t)$ is non-decreasing in x

Exercise 2 from list 1:

exists a unique shock wave? ?



$$x = x_0 + (1-x_0)\, t$$

$t=1$: $\quad x = 1$

At $t=1$ there is a blow-up

Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

$$u_t + \underline{\underline{f'(u)}} \cdot u_x = 0$$

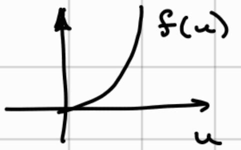
Characteristics are $x = x_0 + f'(u_0(x_0))\, t$

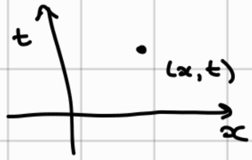
$u \in C^1\ \forall\, t>0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$ if $x_1 < x_2$, otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth f and u_0 are, the solution $u(x,t)$ must become discontinuous

This is a purely non-linear phenomenon!!!

- Assume $f \in C^2$ and $f'' > 0$





$$u_0(x - t f'(u(x,t))) = u(x,t)$$

$$u_t = u_0' \cdot \left(- f'(u(x,t)) - t f''(u(x,t)) \cdot u_t\right)$$

$$u_t\,(1 + t f'' u_0') = - u_0' f'$$

$$u_t = - \frac{u_0' f'}{1 + t f'' u_0'}$$

Analogously, $\quad u_x = \dfrac{u_0'}{1 + t f'' u_0'}$

If $u_0' \geq 0$ (and $f'' > 0$) $\quad u_t$ and u_x stay bounded.

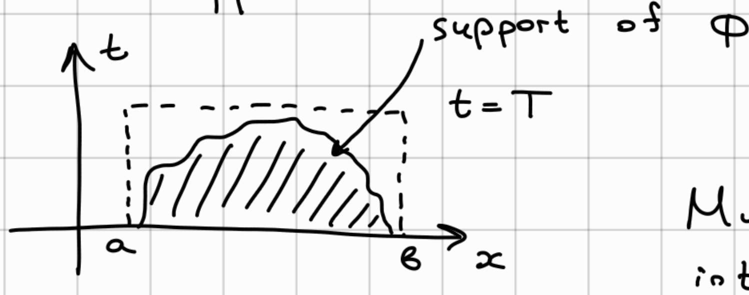
If $u_0' < 0$, then u_t and u_x become unbounded as $1 + t f'' u_0'$ tends to 0.

So we need a notion of <u>weak</u> solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & (1) \\ u\big|_{t=0} = u_0(x) \end{cases}$$

Let u be a classical solution and $\varphi \in C^1$ with compact support:



$\operatorname{supp}(\varphi) \subset D = [a,b] \times [0,T]$

that is φ is zero at $x = a$, $x = b$, $t = T$

Multiply (1) by φ and integrate over $\mathbb{R} \times \mathbb{R}_+$

$$\iint_{t>0} (u_t + f(u)_x)\,\varphi \,dx\,dt = \iint_D (u_t + f(u)_x)\,\varphi\,dx\,dt =$$

$$= \int_a^b \int_0^T (u_t + f(u)_x)\,\varphi\,dx\,dt = \int_a^b u \cdot \varphi \Big|_0^{\cancelto{0}{T}} dx - \int_a^b\int_0^T u \cdot \varphi_t \,dx$$

$$+ \int_0^T \cancelto{0}{f(u)\cdot\varphi\Big|_a^b}\,dt - \int_0^T\int_a^b f(u)\cdot\varphi_x\,dx\,dt =$$

$$= -\int_a^b u_0(x)\varphi(x)\,dx - \int_a^b\int_0^T (u\varphi_t + f(u)\varphi_x)\,dx\,dt$$

$$\Rightarrow \quad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0(x)\varphi(x)\,dx = 0 \qquad (2)$$

$u \in C^1$ and satisfies (1) $\Rightarrow$ u satisfies (2)

But in (2) u not necessarily needs to be C^1!

It can be measurable / bounded.

Definition: A bounded measurable function $u(x,t)$ is called a weak solution of IVP:

$$u_t + f(u)_x = 0, \quad u\big|_{t=0} = u_0(x)$$

↑ bounded / meas.

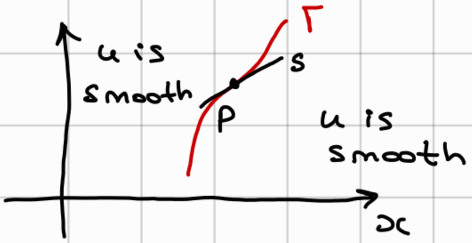
provided that

$$(2) \qquad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0$$

for all $\varphi \in C^1_0$ (φ is C^1 with compact supp)

Rmk: it is clear that if u is in fact C^1, then the original eq. is true: $u_t + f(u)_x = 0$

Lemma (Rankine-Hugoniot condition)



Let Γ be a smooth curve across which u has a jump discontinuity. Take $P \in \Gamma$ and

$u_l = \lim_{(x,t)\to P} u$ from "the left"

$u_r = \lim_{(x,t)\to P} u$ from "the right"

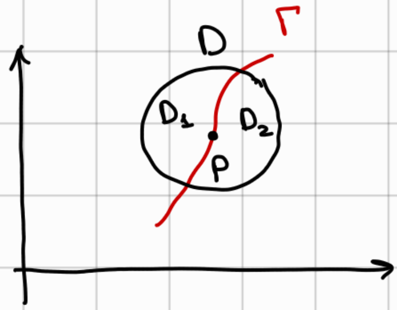
Let the tangent line of Γ at P have the slope $s = \frac{dx}{dt}$. Then: (3) $\boxed{s\cdot(u_l - u_r) = f(u_l) - f(u_r)}$

Often a jump across the shock is denoted:

$[g(u)] = g(u_l) - g(u_r)$, thus we have $s[u] = [f]$

This is called the Rankine-Hugoniot condition

Proof :



Let D be a small ball centered at P and let Γ devide D into two regions D_1 and D_2 (see fig)

Let $\varphi \in C_0^1$ on D and consider

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x)\, dx\, dt = \iint_{D_1} + \iint_{D_2}$$

Divergence theorem : (Green-Gauss theorem)

$$\int_{\partial \Omega} P dx + Q dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x \, dx\, dt = \iint_{D_1} (u\varphi)_t + (f(u)\varphi)_x \, dx\, dt =$$

as $u \in C^1(D_1)$ and $u_t + f(u)_x = 0$

$$= \int f(u)\varphi\, dt - u\varphi\, dx = \int f(u)\varphi\, dt - u\varphi\, dx =$$

t_2, t_1

$$= \int_{t_1}^{t_2} [f(u_e)\varphi(u_e) - u_e \cdot \varphi(u_e) \cdot s]\, dt$$

Similarly,

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x \, dx\, dt = -\int_{t_1}^{t_2} (f(u_r) - su_r)\varphi(u_r)\, dt$$

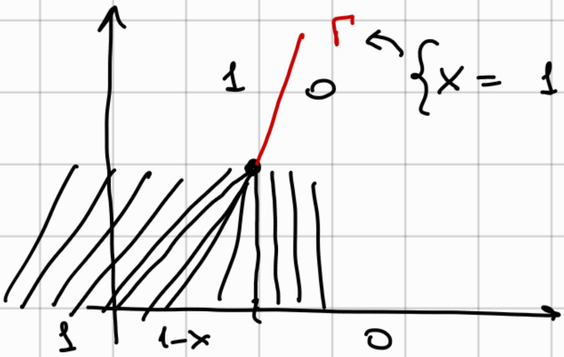
minus because of orientation of ∂D_2 :

Combining together:

$$0 = \int_{t_1}^{t_2} ([f] - s[u])\varphi(u_e)\, dt$$

Since φ was arbitrary, we get $[f] - s[u] = 0$ relation (3).

Example : Burgers eq :



$$s = \frac{\left[\frac{u^2}{2}\right]\Big|_0^1}{[u]\Big|_0^1} = \frac{1}{2}$$

u_e - left state

u_r - right state

in general $s = \frac{u_e + u_r}{2}$

Lecture 7 Scalar conservation law: $\begin{cases} u_t + (f(u))_x = 0 \\ u\big|_{t=0} = u_0(x) \end{cases}$ (*)

$u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f: \mathbb{R}\to\mathbb{R}$, $f \in C^2$, $f''>0$ on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

We want to prove theorems on $\exists$, ! and asymptotic behavior of solutions to (*). From exercise session 1 we remember that we need some extra conditions for this

$\boxed{\text{Thm 1}}$ ($\exists$):

Let $u_0 \in L_\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$, $f''>0$ on $\{u: |u| \le \|u_0\|_\infty\}$.

Then there exists a solution withe the following properties:

(a) $|u(x,t)| \le \|u_0\|_\infty = M$, $(x,t) \in \mathbb{R}\times\mathbb{R}_+$

(b) $\exists\, E>0$ (which depends on M, $\mu = \min\{f''(u): |u|\le M_0\}$ and $A = \max\{|f'(x)| : |u| \le \|u_0\|\}$) ~~such~~

such that $\forall\, a>0, t>0, x\in\mathbb{R}$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

(c) u is stable and depends continuously on u_0: if $v_0 \in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of (*) with initial data v_0, then for $\forall x_1, x_2 \in \mathbb{R}$ with $x_1<x_2$ and $\forall t>0$

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\,dx \le \int_{x_1-At}^{x_2+At} |u_0(x) - v_0(x)|\,dx.$$

~~Remarks: (1)~~

$\boxed{\text{Thm 2}}$ (!):

Let $f\in C^2$, $f''>0$ and let u and v be 2 solutions of (*) satisfying condition (E). Then $u=v$ almost everywhere in $t>0$.

Rmk: we call the solution from Thm 1 (that is satisf. (E) may be there exist more solutions which do not satisfy cond. (E) or (c)

2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm.

3) Cond. **(E)** implies some regularity: u is of locally bounded total variation (for $\forall t$ as a function of x)

Indeed, let c_1 be a constant such that $c_1 > \frac{E}{t}$ and let $v = u - c_1 x$. Then

$$v(x+a,t) - v(x,t) = u(x+a,t) - u(x,t) - c_1 a < a\left(\frac{E}{t} - c_1\right) < 0$$

Thus, v is a non-decreasing function, and v is a function of local bounded total variation.

Since $c_1 x$ is also of bounded total variation, then u is of local bounded total variation.

($\Rightarrow$ countable number of jump discontinuities)

4) finite speed of propagation:

$$v = v_0 \equiv 0 \quad \Rightarrow \quad \int_{x_1}^{x_2} |u(x,t)|\, dx \le \int_{x_1 - At}^{x_2 + At} |u_0(x)|\, dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpetations.

<u>Lemma 1</u>: (a) A smooth solution $u(x,t)$ $(\forall t)$ satisfies condition (E)

(b) If u has a discontinuity at point x_0: (but is smooth to the left and to the right of x_0) and $\lim\limits_{x \to x_0 - 0} u(x,t) = u_L$ and $\lim\limits_{x \to x_0 + 0} u(x,t) = u_R$ and satisfies condition (E) $\Rightarrow u_L > u_R$.

(discontinuities can be only down).

Proof:

⌈▶ (a) Indeed, let us write:

$$u(x,t) = u_0\big(x - t f'(u(x,t))\big)$$

$$u_x = u_0' \cdot (1 - t f'' \cdot u_x) \Rightarrow u_x = \frac{u_0'}{1 + t f'' u_0'}.$$

If u is smooth for $\forall t > 0$, then $u_0' > 0$.

Then $u_x \le \frac{u_0'}{t f'' u_0'} = \frac{E}{t}$ for $E = \frac{1}{\inf f''}$.

Using Lagrange theorem : $\dfrac{u(x+a,t)-u(x,t)}{a} = u_x(\xi,t)$

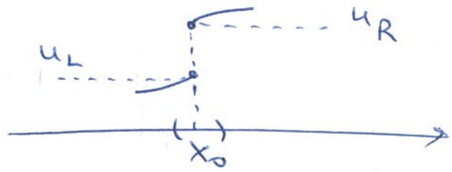
for some $\xi \in [x, x+a)$, and (a) is proved

(b) Either $u_L > u_R$ or $u_L < u_R$ (the case $u_L = u_R$ is not a discontinuity).

- For $u_L < u_R$ the converse of cond. (E) is true:

$\forall E>0 \ \exists x, a>0, t : \quad \dfrac{u(x+a)-u(x)}{a} > \dfrac{E}{t}$.

Indeed, fix E and take small enough neighbourhood of x_0 such that



- for $x \in (x_0-\delta, x_0)$ $\quad |u-u_L| \leq \varepsilon = \dfrac{u_R-u_L}{4}$
- for $x \in (x_0, x_0+\delta)$ $\quad |u-u_R| \leq \varepsilon = \dfrac{u_R-u_L}{4}$.

This means that for $\forall\, x_1 \in (x_0-\delta, x_0)$ and $x_2 \in (x_0, x_0+\delta)$

$$u(x_2)-u(x_1) \geq \frac{u_R-u_L}{2}.$$

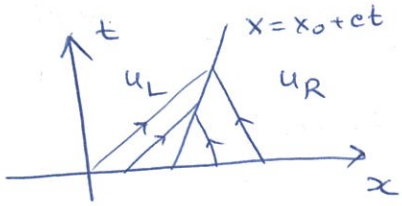
Fix t and take small a :

$x_2 - x_1 = a$
$x_1 \in (x_0-\delta, x_0)$
$x_2 \in (x_0, x_0+\delta)$

$$\frac{u(x_2)-u(x_1)}{a} \geq \frac{u_R-u_L}{2a} = \frac{E}{t}.$$

- For $u_L > u_R$ $\quad \dfrac{u(x+a)-u(x)}{a} \leq 0$, thus $\forall E>0$ is ok ■

Lemma 2 (Remark) : u satisfies condition (E) and is a shock wave solution $u = \begin{cases} u_L, & x<ct \\ u_R, & x>ct \end{cases}$



then $\quad f'(u_L) > c > f'(u_R)$ $\quad$ (Lax condition)

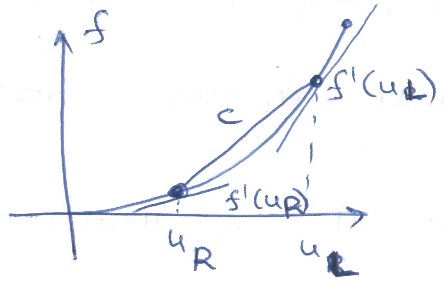
" Characteristics come to the line of discontinuity "

The converse situation corresponds to the case

in some sense where " the information" appears on the discontinuity - which is not

We will generalize the Lax condition to the case of systems.

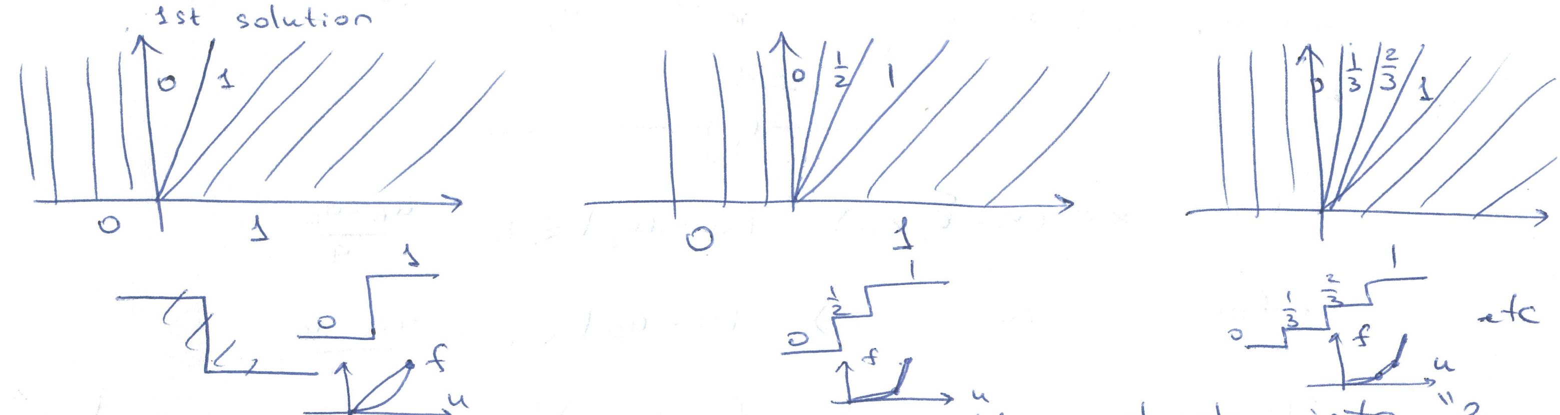
Indeed, $f'' > 0 \Rightarrow$ (see picture)



$$f'(u_L) > c = \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R)$$

Remark (on Liu criterion) "internal stability of shock"

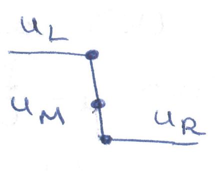
Remember the situation with Burgers equation:



In some sense if we divide the shock into "2 smaller" shocks, they could have a tendency of eigher gluing into 1 shock (some kind of stability) or going further one from another (instability)

Condition (E) $\Rightarrow$ this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L, u_R) = \frac{f(u_R) - f(u_L)}{u_R - u_L} \leq c(u_L, u_M) = \frac{f(u_L) - f(u_M)}{u_L - u_M}$$



$$\rotatebox{90}{$\leq$} \qquad c(u_M, u_R) = \frac{f(u_R) - f(u_M)}{u_R - u_M} \qquad \forall u_M \in (\min(u_L, u_R), \max(u_L, u_R))$$

If $\begin{cases} u_M \to u_L \\ u_M \to u_R \end{cases}$ we have Lax condition.

Vanishing viscosity criterion for shock waves.

- We think of equation $u_t + (f(u))_x = 0$ as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \underbrace{\varepsilon u_{xx}}_{\text{small regularizing term}}, \quad \varepsilon > 0 \qquad (P)$$

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks)

Rmk 2: equation (P) is a combination of 2 "opposite" effects

→ $u_t + (f(u))_x = 0$ ⇝ creates shocks:

→ $u_t = \varepsilon u_{xx}$ ⇝ "smooths":

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

$$u(x,t) = v(x-ct)$$

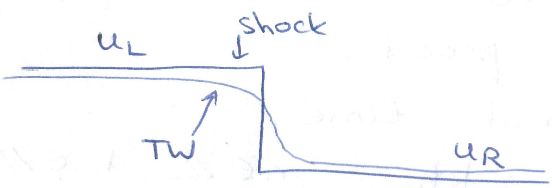
for $c \in \mathbb{R}$ and v - some smooth profile.

They look like "smoothed" shocks!!!

This motivates the following definition:

Def 1 (vanishing viscosity criterion for shock waves): A shock wave is an entropy solution if is a limit in L^1_{loc} of a travelling wave solution of (P) as $\varepsilon \to 0$.

$f \in C^2$, $f'' > 0$



Lemma: a shock wave is an entropy solution in sense of def 1, iff $u_L > u_R$.

Proof: Let's look for travelling wave solutions for eq. (P): $v\left(\frac{x-ct}{\varepsilon}\right)$: $v(-\infty) = u_L$, $v(+\infty) = u_$

$$-cv' + (f(v))' = \varepsilon v''$$

Integrate $\int_{-\infty}^{+\infty}$: $-c(u_R - u_L) + f(u_R) - f(u_L) =$

Interesting feature: it is exactly RH condition

OK, let us integrate $\int_{-\infty}^{\xi}$: $-c(v(\xi) - u_L) + (f(v(\xi)) - f(u_$
$= v'(\xi)$

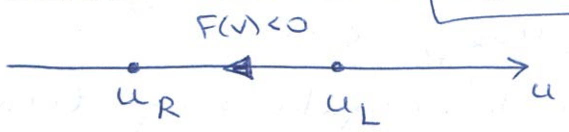
ODE) $v' = f(v) - f(u_L) - c(v - u_L) = F(v)$

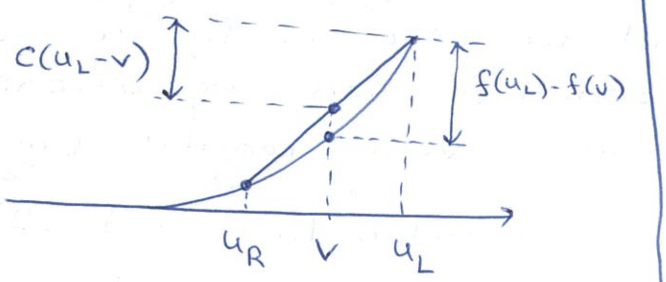
Note that RHS $F(u_L) = 0$ and $F(u_R) = 0$ (due to RH!)

Thus u_L and u_R are two fixed points of this ODE

Consider 2 cases: $\boxed{u_L > u_R}$ and $\boxed{u_L < u_R}$.

Case $u_L > u_R$:

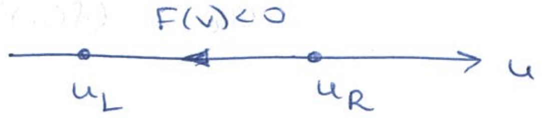


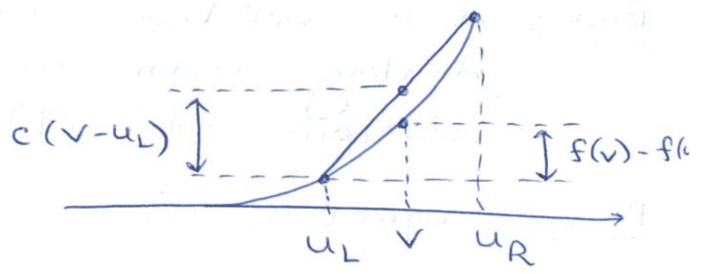


In this case: $F(v) < 0$ $\forall v \in (u_R, u_L)$

And there exists a solution v of ODE: $v(-\infty) = u_L$; $v(+\infty) = u_R$

Case $u_L < u_R$:





In this case: $F(v) < 0$ $\forall v \in (u_L, u_R)$

And there DOES NOT exist a solution v of ODE $v(-\infty) = u_L$; $v(+\infty) = u_R$ ■

Lecture 8: Scalar conservation law:

$$\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases} \quad (*)$$

- $u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f: \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f''>0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\, dx\, dt + \int_{t=0} u_0 \varphi\, dx = 0 \quad (**)$$

for every test function $\varphi \in C_0^1$.

Define $M := \|u_0\|_\infty$, $A := \max_{|u|\le M} |f'(u)|$, $\mu := \min_{|u|\le M} f''(u)$

Today we will start proving theorem on existence.

<u>Thm 1</u> ($\exists$): Let $u_0 \in L_\infty(\mathbb{R})$; $f \in C^2(\mathbb{R})$, $f''>0$ on $\{u : |u| \le M\}$

There exists a solution with the following properties

(a) $|u(x,t)| \le M$, $(x,t) \in \mathbb{R}\times\mathbb{R}_+$

(b) $\exists E = E(M,\mu,A) > 0$ such that $\forall a>0$, $\forall t>0$, $x \in \mathbb{R}$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t} \quad (E)$$

"entropy" cond.

(c) u is stable and depends continuously on u_0: if $v_0 \in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of $(*)$ with initial data v_0, then for $\forall x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and $t>0$

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\, dx \le \int_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)|\, dx \quad (S)$$

"stability"

How to prove this theorem?

There exist (at least) 5 approaches:

(a) Calculus of variations and Hamilton-Jacobi theory

(b) Vanishing viscosity method

(c) Non-linear semigroup theory

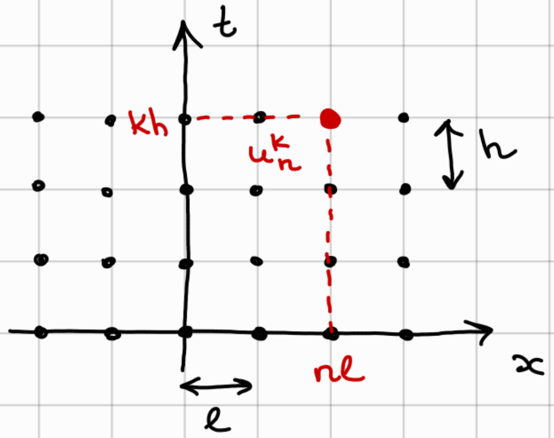
(d) Method of characteristics

(e) Finite-difference method

We will follow Smoller (Chapter 16) and use (e).

Here is the scheme of the proof:

<u>Step 1</u>: discretization in space and time



$x_n = n\ell, \quad n \in \mathbb{Z} \qquad \ell = \Delta x > 0$

$t_k = kh, \quad k \in \mathbb{N} \cup \{0\} \qquad h = \Delta t > 0$

$u_n^k = u(n\ell, kh)$

Consider a finite-difference (explicit) scheme:

$$(D) \quad \boxed{u_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell} \cdot \left(f(u_{n+1}^k) - f(u_{n-1}^k) \right)}$$

$u_n^0 = u_0(n\ell), \quad n \in \mathbb{Z}$

In what follows we will always assume:

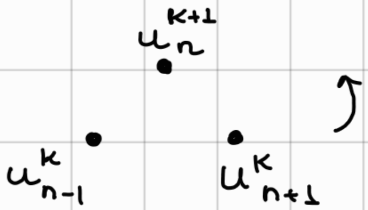
$$\boxed{\frac{Ah}{\ell} \leq 1} \qquad \text{(CFL condition)}$$

Courant-Friedrichs-Lewy

It is important for the stability of the numerical scheme and tells that the time step h should be small enough.

First, we will formulate and prove some properties of solutions u_n^k of a discrete eq. (D):

(1a) solution exists (evident!)



(1b) if $|u_n^0| \leq M$, then $|u_n^k| \leq M \quad \forall k \in \mathbb{N}$

(boundedness)

(1c) $\exists E = E(M, A, \mu) > 0$: $\dfrac{u_n^k - u_{n-2}^k}{2\ell} \le \dfrac{E}{kh}$ (E-disc)

discrete entropy condition

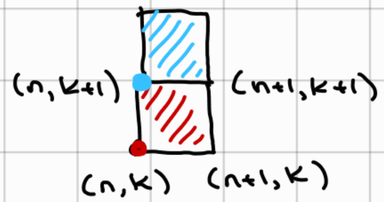
NB: the discrete entropy condition is a natural consequence of a finite difference approximation (D).

(1d) local bounded variation: $\forall X > 0$ and $kh > \alpha > 0$ $\exists\, c(X, \alpha)$ (but independent of h and ℓ):

$$\sum_{|n| \le X/\ell} |u_{n+2}^k - u_n^k| \le C$$ and some other...

Step 2: We will prove convergence as $h, \ell \to 0$.

Consider $U_{h,\ell}(x, t) = u_n^k$ if
$n\ell \le x \le (n+1)\ell$
$kh \le t \le (k+1)h$



We will prove that there exists subsequence U_{h_i,ℓ_i} of $U_{h,\ell}$ such that $U_{h_i,\ell_i} \to u(x,t)$ - some measurable function

Step 3: We will prove that this limiting function satisfies integral equality (**) and all properties of theorem on $\exists$.

Proof of the theorem 1.

Lemma 1 (boundedness of u_n^k): $|u_n^k| \le M$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$

This is an exercise 2 from list 3.

Lemma 2 (discrete entropy condition)

If $c = \min\left(\frac{\mu}{2}, \frac{A}{4M}\right)$, then

$$\frac{u_n^k - u_{n-2}^k}{2\ell} \le \frac{E}{kh} \quad \text{where } E = \frac{1}{c}.$$

Proof:

▶ Let $z_n^k = \dfrac{u_n^k - u_{n-2}^k}{2\ell}$ and first let us prove some recurrent relation for z_n^{k+1} of the form

" $z_n^{k+1} = A\, z_{n+1}^k + B\, z_{n-1}^k + C$ "

$$z_n^{k+1} = \frac{1}{2}\left[z_{n+1}^k + z_{n-1}^k \right] - \frac{h}{(2\ell)^2}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$
$$+ \frac{h}{(2\ell)^2}\left(f(u_{n-1}^k) - f(u_{n-3}^k)\right)$$

Notice that due to $f \in C^2$ we can write

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k)\,(u_{n+1}^k - u_{n-1}^k) + f''(\theta_1)\frac{(u_{n+1}^k - u_{n-1}^k)^2}{2}$$

for some θ_1 between u_{n+1}^k and u_{n-1}^k

$$= f(u_{n-1}^k) + f'(u_{n-1}^k)\cdot 2\ell \cdot z_{n+1}^k + f''(\theta_1)\cdot\frac{(2\ell)^2}{2}\left(z_{n+1}^k\right)^2$$

Analogously,

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k)\cdot 2\ell\cdot z_{n-1}^k + f''(\theta_2)\frac{(2\ell)^2}{2}\left(z_{n-1}^k\right)^2$$

Thus,

$$z_n^{k+1} = z_{n+1}^k\cdot\Big[\overbrace{\frac{1}{2} - \frac{h}{2\ell} f'(u_{n-1}^k)}^{A}\Big] + z_{n-1}^k\Big[\overbrace{\frac{1}{2} + \frac{h}{2\ell} f'(u_{n-1}^k)}^{B}\Big]$$
$$- \frac{h}{2}\cdot\left[f''(\theta_1)\cdot\left(z_{n+1}^k\right)^2 + f''(\theta_2)\cdot\left(z_{n-1}^k\right)^2\right]$$

Note that $A + B = 1$ and $A, B \geq 0$

Define $\tilde{z}_n^k = \max\{ z_{n-1}^k, z_{n+1}^k, 0\}$

If $\tilde{z}_n^k = 0 \quad \forall n$, then (E-disc) is true since

$$z_n^k \leq \tilde{z}_n^k = 0 \leq \frac{E}{kh} \quad \forall E > 0.$$

Thus, w.l.o.g. we can assume $\tilde{z}_n^k \neq 0$. Suppose $\tilde{z}_n^k = z_{n+1}^k$ (the other case is similar)

$$z_n^{k+1} \leq z_{n+1}^k\cdot\left[A + B\right] - h\frac{\mu}{2}\left(z_{n+1}^k\right)^2 \leq$$

$$\leq \tilde{z}_n^k + h c\left(\tilde{z}_n^k\right)^2.$$

Notice that

$$z_n^k \leq |z_n^k| \leq \frac{M}{\ell} \underset{\substack{\uparrow \\ \text{CFL}}}{\leq} \frac{M}{Ah} \underset{\substack{\uparrow \\ c \leq \frac{A}{4M}}}{\leq} \frac{M}{h}\cdot\frac{1}{4Mc} = \frac{1}{4ch}$$

Let $M^k = \max_{n\in\mathbb{Z}}\{\tilde{z}_n^k\} \geq 0.$

Let $\varphi(y) = y - c\cdot h\cdot y^2$. Since $\varphi' = 1 - 2chy$, φ is increasing if $y \leq \frac{1}{2ch}$. But we have

$$\tilde{z}_n^k \leq M^k \leq \frac{1}{4ch} < \frac{1}{2ch}.$$

So that $\varphi(\tilde{z}_n^k) \leq \varphi(M^k)$ and we have

$$\tilde{z}_n^k - ch(\tilde{z}_n^k)^2 \leq M^k - ch(M^k)^2$$

Thus, $z_n^{k+1} \leq M^k - ch(M^k)^2 \quad \forall n \in \mathbb{Z}$.

It follows that

$$\boxed{M^{k+1} \leq M^k - ch(M^k)^2} \quad (M)$$

<u>Claim</u>: $M^k \leq \frac{1}{chk + 1/M^0}$.

Suppose we have proven claim. Let us see how it helps to prove lemma 2. Indeed,

$$z_n^k \leq M^k \leq \frac{1}{chk + 1/M^0} \leq \frac{1}{chk} = \frac{E}{hk}, \quad E = \frac{1}{c}.$$

Proof of claim: first – intuition why such estimate could be true

Inequality (M) for M^k is a discrete analog of ODinEquality: $\varphi' \leq -ch\varphi^2$

if it was an equality $\varphi' = -ch\varphi^2$, then the solution is:

$$\frac{d\varphi}{\varphi^2} = -ch\,dt$$

$$-\frac{1}{\varphi} = -cht + C_1$$

$$\varphi(t) = \frac{1}{cht - C_1}$$ and with ic $\varphi(0) = \varphi_0$

we will have $\varphi(t) = \frac{1}{cht + 1/\varphi_0}$.

So one can try to prove $\varphi(t) \leq \frac{1}{cht + 1/\varphi_0}$.

Second, let us make the formal proof. We will do it by induction.

Base: $k=0$: – clear: $M^0 \le \frac{1}{1/\mu^0} = M^0$.

$k>0$: suppose that
$$M^k \le \frac{1}{chk + 1/\mu^0}$$
and we want to prove that
$$M^{k+1} \le \frac{1}{ch(k+1) + 1/\mu^0}$$

We have: $\frac{1}{M^k} \ge chk + \frac{1}{M^0}$, so
$$1 - chM^k \ge 1 - chk\,M^k \ge \frac{M^k}{M^0} \ge 0.$$

Thus $1 - (ch\,M^k)^2 \ge 0$.

We have $M^{k+1} \le M^k(1 - chM^k)$, so that
$$\frac{M^{k+1}}{1 - chM^k} \le M^k \le \frac{M^k}{1 - (chM^k)^2}$$

and thus
$$M^{k+1} \le \frac{M^k}{1 + chM^k} = \frac{1}{ch + 1/M^k} \le$$
$$\le \frac{1}{ch(k+1) + 1/\mu^0} \qquad \text{q.e.d.} \quad \blacksquare$$

Lemma (space estimate): For any $X>0$ and $kh \ge \alpha > 0$, there is a constant $C = C(X, \alpha, M)$ (but independent on h, ℓ) such that:
$$\sum_{|n| \le X/\ell} |u^k_{n+2} - u^k_n| \le C$$

Proof:

Set $v^k_n = u^k_n - c_1 n \ell$, where c_1 is chosen so large that $E/\alpha < c_1$. Then
$$v^k_{n+2} - v^k_n = u^k_{n+2} - u^k_n - 2c_1\ell \le \frac{2\ell E}{kh} - 2c_1\ell \le$$
$$\le 2\ell\left(\frac{E}{\alpha} - c_1\right) < 0,$$
so v^k_n is decreas. in n

Thus
$$\sum_{|n| \le X/\ell} |u^k_{n+2} - u^k_n| \le \sum |v^k_{n+2} - v^k_n| + \sum 2c_1\ell =$$
$$= \underbrace{-\sum (v^k_{n+2} - v^k_n)}_{\text{telescopic sum}} + 2c_1\ell\left(\frac{2X}{\ell} + 1\right) \le 4M + 2c_1X + c_2X$$
$$\le 4(M + c_1X) \qquad \text{q.e.d.} \quad \blacksquare$$

Lecture 9: We continue proving theorem on existence of entropy solution for scalar cons law.

Lemma 4 (time estimate – u_n^k are L^1 locally Lipshits in k)

If $h/e \geq \delta > 0$ and $h, e \leq 1$, then exists $L > 0$ (independent of h, e) such that

if $k > p$, where $(k-p)$ is even and $ph \geq \alpha > 0$, then

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| \, e \leq L(k-p)h$$

A similar estimate holds if $(k-p)$ is odd.

Proof:

Let us express u_n^k in terms of u_n^p where $(k-p)$ is even.

$$u_n^k = \frac{1}{2}\left(u_{n+1}^{k-1} + u_{n-1}^{k-1}\right) - \frac{h}{2e} f'(\theta)\left(u_{n+1}^{k-1} - u_{n-1}^{k-1}\right) =$$

$$= u_{n+1}^{k-1}\left(\frac{1}{2} - \frac{h}{2e} f'(\theta)\right) + u_{n-1}^{k-1}\left(\frac{1}{2} + \frac{h}{2e} f'(\theta)\right)$$

or $u_n^k = a_{n+1}^{k-1} u_{n+1}^{k-1} + a_{n+1}^{k-1} u_{n+1}^{k-1}$, where

$a_{n+1}^{k-1} + a_{n-1}^{k-1} = 1$ and $a_{n+1}^{k-1}, a_{n-1}^{k-1} \geq 0$.

Applying this to u_{n-1}^k and u_{n+1}^k gives a formula:

$$u_n^{k+1} = A u_{n+2}^{k-1} + B u_n^{k-1} + C u_{n-2}^{k-1}$$

where $A, B, C \geq 0$, $A + B + C = 1$.

Hence, $|u_n^{k+1} - u_n^{k-1}| \leq A|u_{n+2}^{k-1} - u_n^{k-1}| + C|u_{n-2}^{k-1} - u_n^{k-1}|$

Multiplying this by $\Delta x = e$ and summing, we get:

$$\sum_{|n| \leq X/e} |u_n^{k+1} - u_n^{k-1}| \, \Delta x \underset{\text{lemma 3}}{\leq} c\, \Delta x$$

Now if $(k-p)$ is even, we can do this operation several times and using the triangle inequality, we get:

$$\sum_{|n| \leq X/e} |u_n^k - u_n^p| \, \Delta x \leq \sum_{i=p}^{k-2} \sum_{|n| \leq X/e} |u_n^{i+2} - u_n^i| \, \Delta x \leq (k-p) c\, \Delta x \leq$$

$$L \leq \frac{\Delta t}{\delta}(k-p)c = L(k-p)h \quad \text{for } L = \frac{c}{\delta}, \; h = \Delta t \quad \blacksquare$$

Lemma 5 (stability): Let u_n^k and v_n^k be solutions to the finite-difference scheme (D) corresponding to the initial conditions u_n^0 and v_n^0, respectively, where

$$\sup_{n\in\mathbb{Z}} |u_n^0| \leq M \quad \text{and} \quad \sup_{n\in\mathbb{Z}} |v_n^0| \leq M$$

Then, if $k>0$,

$$\sum_{|n|\leq N} |u_n^k - v_n^k| \cdot \ell \leq \sum_{|n|\leq N+k} |u_n^0 - v_n^0| \cdot \ell$$

Proof:

Γ $w_n^k = u_n^k - v_n^k$. From (D) we have

$$w_n^{k+1} = u_n^{k+1} - v_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$

$$- \frac{v_{n+1}^k + v_{n-1}^k}{2} + \frac{h}{2\ell}\left(f(v_{n+1}^k) - f(v_{n-1}^k)\right) =$$

$$= \frac{w_{n+1}^k + w_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(v_{n+1}^k)\right)$$

$$+ \frac{h}{2\ell}\left(f(u_{n-1}^k) - f(v_{n-1}^k)\right) =$$

$$= w_{n+1}^k \underbrace{\overbrace{\left[\frac{1}{2} - \frac{h}{2\ell} f'(Q_1)\right]}^{A_{n+1}^k}}_{\geq 0 \text{ due to CFL}} + w_{n-1}^k \underbrace{\overbrace{\left[\frac{1}{2} + \frac{h}{2\ell} f'(Q_2)\right]}^{B_{n+1}^k}}_{\geq 0}$$

Now proceed by induction.

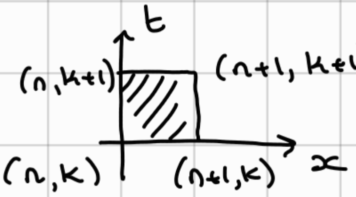
$$\sum_{|n|\leq N} |w_n^{k+1}| \leq \sum_{|n|\leq N} |w_{n+1}^k| \cdot A_{n+1}^k + \sum_{|n|\leq N} |w_{n-1}^k| \cdot B_{n+1}^k =$$

$$= \sum_{1-N}^{N+1} |w_m^k| A_m^k + \sum_{-1-N}^{N-1} |w_m^k| \cdot B_m^k \leq$$

$$\leq \sum_{|m|\leq N+1} |w_m^k| A_m^k + \sum_{|m|\leq N+1} |w_m^k| \cdot B_m^k \leq \sum_{|m|\leq N+1} |w_m^k| \quad \blacksquare$$

q.e.d.

Step 2: Rather than define u_n^k in mesh points let us continue u_n^k as a piece wise constant function in the upper half plane.



$$\mathcal{U}_{h,\ell}(x,t) = u_n^k \quad \text{if} \quad \begin{array}{l} n\ell \le x \le (n+1)\ell \\ kh \le t \le (k+1)h \end{array}$$

So we have a family of functions $\{\mathcal{U}_{h,\ell}\}$ and would like to choose a convergent subsequence $\mathcal{U}_{h_i,\ell_i}$ as $h_i, \ell_i \to 0$ $i \to \infty$.

Lemma 6 (convergence: the set of functions $\{\mathcal{U}_{h,\ell}\}$ is compact in the topology of L_1-convergence on compacta)

There exists a subsequence $\{\mathcal{U}_{h_i,\ell_i}\}$ $i \in \mathbb{N}$ which converges to a measurable function $u(x,t)$ in the sense that for $\forall X>0, t>0, T>0$ both

$$\int_{|x|\le X} |\mathcal{U}_{h_i,\ell_i}(x,t) - u(x,t)|\, dx \to 0 \quad \text{as } h_i, \ell_i \to 0$$

and

$$\int_0^T \int_{|x|\le X} |\mathcal{U}_{h_i\ell_i}(x,t) - u(x,t)|\, dx\, dt \to 0 .$$

Furthermore, the function $u(x,t)$ satisfies:

(a) $\sup\limits_{\substack{x\in\mathbb{R}\\ t>0}} |u(x,t)| \le M$; (b) inequality (S) (stability)

Proof:

▶ First, take $t = \text{const}$ and consider $\mathcal{U}_{h,\ell}(x,t)$ as functions of x. By Lemma 1 and Lemma 3 the set of functions $\{\mathcal{U}_{h,\ell}\}$ is bounded and have uniformly bounded total variation on each bounded interval in x.

__Helly's theorem__ (simple version):

A uniform bounded sequence of monotone, real functions admits a convergent subsequence.

__Helly's theorem__ (generalized version):

A uniform bounded sequence of BV_{loc} (locally of bounded variation) real functions admits a convergent subsequence on every compact set.

__Rmk__: a function of BV_{loc} can be written as a sum of increasing and decreasing functions (on each compact interval). This is why the generalized version of the Helly's theorem is true.

So by Helly's theorem on each interval we have a convergent subsequence $U'_{h,e}$.

By a standard diagonal process we can construct a subsequence $\{U''_{h,e}\}$ from $\{U'_{h,e}\}$ which converges at every $x \in \mathbb{R}$ for this particular $t = const > 0$.

Second, take $\{t_m\}_{m=1}^{\infty}$ - a countable and dense subset of $(0,T)$, e.g. $\mathbb{Q} \cap (0,T)$.

For $t = t_1$ we have $\{U^1_{h_i, e_i}\}$ a convergent subsequence.

For $t = t_2$ take a convergent sub. $\{U^2_{h_i, e_i}\}$ from $\{U^1_{h_i, e_i}\}$ etc. So we have:

$$
\begin{array}{lllll}
t=t_1: & U^1_{h_1,e_1} & U^1_{h_2,e_2} & U^1_{h_3,e_3} & U^1_{h_4,e_4} \\
t=t_2: & U^2_{h_1,e_1} & U^2_{h_2,e_2} & U^2_{h_3,e_3} & U^2_{h_4,e_4} \\
t=t_3: & U^3_{h_1,e_1} & U^3_{h_2,e_2} & U^3_{h_3,e_3} & U^3_{h_4,e_4} \\
t=t_4: & U^4_{h_1,e_1} & U^4_{h_2,e_2} & U^4_{h_3,e_3} & U^4_{h_4,e_4} \\
 & & & & \ddots
\end{array}
$$

By a standard diagonal process, we can choose a subsequence U_{h_i, e_i} which converges for all $\{t_m\}_{m=1}^{\infty}$ and all $x \in \mathbb{R}$.

Third, we want to show that there is a convergence for all $t \in (0,T)$. So that in the limit we indeed obtain a function defined in the strip $0 < t < T$

Let $U_i = U_{l_i, h_i}$ and we want to show that

$$I_{ij} = \int_{-X}^{X} | U_i(x,t) - U_j(x,t) | \, dx \longrightarrow 0 \quad \forall i,j \to \infty$$

i.e. that $\{U_i\}$ is a Cauchy sequence in $L_1(|x| \le X)$

For $t \in (0,T)$ we find a subsequence $\{t_{m_s}\} \subset \{t_m\}$ such that $t_{m_s} \to t$ as $s \to \infty$. Let $\tau_s = t_{m_s}$. Then

$$I_{ij}(t) \le \int_{-X}^{X} | U_i(x,t) - U_i(x,\tau_s) | \, dx + \int_{-X}^{X} | U_i(x,\tau_s) - U_j(x,\tau_s) |$$

$$+ \int_{-X}^{X} | U_j(x,t) - U_j(x,\tau_s) | \, dx =: I_1 + I_2 + I_3$$

For $t = \tau_s$ we have a convergence of U_i, thus for s large enough we have $I_2 < \varepsilon/3$

Let's estimate I_1 :

$$I_1 = \int_{-X}^{X} | U_i(x, [\tfrac{t}{h_i}] h_i) - U_i(x, [\tfrac{\tau_s}{h_i}] h_i) | =$$

$$= \sum_{|n| < \frac{X}{l_i} + 1} \int_{n l_i}^{(n+1) l_i} | U_i(x, [\tfrac{t}{h_i}] h_i) - U_i(x, [\tfrac{\tau_s}{h_i}] h_i) | \, dx =$$

$$= \sum_{|n| < \frac{X}{l_i} + 1} | u_n^{[t/h_i]} - u_n^{[\tau_s/h_i]} | \, l_i \underset{\substack{\uparrow \\ \text{Lemma 4}}}{\le} L h_i \left| [\tfrac{t}{h_i}] - [\tfrac{\tau_s}{h_i}] \right|$$

$$\le L | t - \tau_s | < \frac{\varepsilon}{3} \quad \text{for } s \text{ large enough.}$$

Analogously, $I_3 < \frac{\varepsilon}{3}$. Thus $I_{ij} \le 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

We have proved pointwise limit for every $t \in (0,T)$, that is $\exists \, u(x,t) \in L_1(|x| \le X)$ (in part, measurable)

Fourth, let us show that $I_{ij} \to 0$ uniformly in t, $0 < \tau \le t \le T$. Indeed, fix $\varepsilon > 0$. Choose finite subset $\mathcal{F} \subset \{t_m\}$ such that if $0 \le t \le T$ there is a $t_m \in \mathcal{F}$ such that $L(t - t_m) < \frac{\varepsilon}{3}$. Then we choose i, j so large that $I_2 < \frac{\varepsilon}{3}$ for all $t_m \in \mathcal{F}$ (it is possible because $\mathcal{F}$ is finite). This reasoning gives us the desired uniformity in t

Fifth, using uniform convergence, we have

$$\forall \tau \in (0, T] \qquad \int_\tau^T I_{ij}\, dt \to 0 .$$

Now we write $\int_0^T = \int_0^\tau + \int_\tau^T$:

$$\int_0^T \int_{-X}^X |U_i - U_j|\, dx\, dt = \underbrace{\int_0^\tau \int_{-X}^X |U_i - U_j|}_{< \varepsilon/2 \text{ if } 8MX\tau < \varepsilon} + \overbrace{\int_\tau^T \int_{-X}^X |U_i - U_j|}^{< \frac{\varepsilon}{2} \text{ for } i,j \text{ suffic. large}} < \varepsilon$$

That means

$$\int_0^T I_{ij}\, dt \longrightarrow 0 \quad \text{as} \quad i,j \to +\infty .$$

Sixth, since local convergence in L_1 implies poinwise convergence a.e. of a subsequence, we see

$$|U_i| \le M \implies |u| \le M$$

and Lemma 5 $\Rightarrow$ (S) ∎

Step 3: Let us show that the limiting function $u(x,t)$, indeed, satisfies the properties from thm 1.

Lemma 7 (entropy inequality): u satisfies (E).

Proof:

It is sufficient to show that if $(x_1-x_2) > 2l_i$ and $t > h_i$ then

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1-x_2} < \frac{2E}{t-h_i}.$$

Let $x_1 > x_2$ and note that

$$U_i(x_j,t) = U_i\left(x_j - \eta_j, \left[\frac{t}{h_i}\right] h_i\right) \quad j=1,2$$

for some $0 \le \eta_j < l_j$. Thus,

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1-x_2} = \frac{1}{x_1-x_2} \sum (u_n^k - u_{n-2}^k) \quad \text{for } k = \left[\frac{t}{h_i}\right]$$

over all integers in the interval $[x_2-\eta_2, x_1-\eta_1]$

Using Lemma 2, we have

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1-x_2} \le \frac{E(x_1-\eta_1-x_2+\eta_2)}{\left[\frac{t}{h_i}\right] h_i (x_1-x_2)} \le \frac{E(x_1-\eta_1-x_2+\eta_2)}{(t-h_i)(x_1-x_2)} =$$

$$= \frac{E}{t-h_i} + \frac{E\overbrace{(\eta_2-\eta_1)}^{<l_i}}{(t-h_i)\underbrace{(x_1-x_2)}_{>2l_i}} < \frac{2E}{t-h_i} \blacksquare$$

Lecture 10 : Let's finish proving theorem on $\exists$ of entropy solution

Reminder : Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

  $f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f'' > 0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense :

$$\iint\limits_{t>0} [u\varphi_t + f(u)\varphi_x]\, dx\, dt + \int\limits_{t=0} u_0 \varphi\, dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

Lemma 8 (last lemma)

Let v_i be a convergent subsequence from Lemma 6.
We know that $v_i \to u(x,t)$, $i \to +\infty$, and $\forall X \in \mathbb{R}$

$$\int_{-X}^{X} |v_i(x,0) - u_0(x)|\, dx \to 0.$$

Then u satisfies $(**)$, i.e. u is a weak solution of $(*)$.

Proof.

$\blacktriangleright$ Rewrite (D) in such a form:

$$\frac{u_n^{k+1} - u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2\ell^2} \cdot \frac{\ell^2}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2\ell} = 0$$

Multiply this equality by $\varphi_n^k = \varphi(n\ell, kh)$ and get

$$\frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} - u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} + \frac{\ell^2}{h} \cdot u_n^k \cdot \frac{2\varphi_n^k - \varphi_{n+1}^k - \varphi_{n-1}^k}{\ell^2}$$

$$+ \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} + \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} +$$

$$+ \frac{\varphi_{n+1}^k f(u_{n+1}^k) - \varphi_{n-1}^k f(u_{n-1}^k)}{2\ell} - f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2\ell}$$

$$- f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2\ell} = 0$$

Since $\varphi \in C^3_0$ has compact support, we may assume $\varphi_n^k = 0$ if $k \geq [\frac{T}{h}]$

Multiply this equality by hl and sum over $n \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$.

- $\displaystyle\sum_{k,n} \frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} = -\sum_n \varphi_n^0 u_n^0$ (telescopic sum)
- $\displaystyle\sum_{k,n} \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} = 0$ and $\displaystyle\sum_{k,n} \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} = 0$

Thus,

$$-h \sum_n \varphi_n^0 u_n^0 + hl \left[\sum_{k,n} \left[-u_n^{k+1} \frac{\varphi_n^{k+1} - \varphi_n^k}{h} - \frac{l^2}{2h} \overset{u_n^k}{\vee} \frac{\varphi_{n+1}^k + \varphi_{n-1}^k - 2\varphi_n^k}{2l} \right] \right.$$
$$\left. - \sum_{k,n} f(u_{n+1}^k) \frac{\varphi_{n+1}^k - \varphi_n^k}{2l} - \sum_{k,n} f(u_{n-1}^k) \frac{\varphi_n^k - \varphi_{n-1}^k}{2l} \right] = 0$$

Instead of a sum for u_n^k we can write integral for $U_{h,l}$

$$-\int_{t=0} U_{h,l}\varphi + \delta_1 - \iint_{t\geq 0} U_{h,l}\varphi_t + \delta_2 - \frac{l^2}{2h}\iint_{t\geq 0} U_{h,l}\varphi_{xx}$$
$$+\delta_3 - \iint_{t\geq 0} f(U_{h,l})\varphi_x + \delta_4 = 0$$

where $\delta_i \to 0$ as $h, l \to 0$. Replace $U_{h,l}$ by U_i:

$$-\int_{t=0} U_i\varphi - \iint_{t\geq 0} U_i\varphi_t - \frac{l_i^2}{2h_i}\iint_{t\geq 0} U_i\varphi_{xx} - \iint_{t\geq 0} f(U_i)\varphi_x = \delta(h_i, l_i)$$
$$\delta(h_i,l_i) \xrightarrow{i\to\infty} 0$$

$l_i \to 0$, $\frac{l_i}{h_i}$ is bounded ; $\frac{l_i^2}{h_i} \to 0$; $U_i \to u$ in L^1-loc

$$\Rightarrow \iint_{t\geq 0} U_i\varphi_t - \frac{l_i^2}{2h_i}\iint_{t\geq 0} U_i\varphi_{xx} \to \iint_{t\geq 0} u\varphi_t$$

By choice of initial values: $\int\limits_{t=0} U_i \varphi \to \int\limits_{t=0} u_0 \varphi$

Also, $\left| \iint\limits_{t\geq 0} (f(U_i) - f(u)) \varphi_x \right| \leq \|\varphi_x\|_\infty \iint\limits_{D:\, \varphi \neq 0} |f(U_i) - f(u)|$

$$\leq \|\varphi_x\|_\infty \iint\limits_{D:\varphi\neq 0} |f'(\xi)| \cdot |U_i - u| \to 0$$

And we have: $$\iint\limits_{t\geq 0} f(U_i)\varphi_x \to \iint\limits_{t\geq 0} f(u)\varphi_x .$$

We have proved $(**)$ for $\forall \varphi \in C_0^3$.
$C_0^3 \subset C_0^1$ is a dense subset, then $(**)$ are also true for $\varphi \in C_0^1$. ■

Now, let's prove the theorem on uniqueness.

Thm 2 (!): Let $f \in C^2$, $f'' > 0$.
Let u, v be 2 solutions of $(**)$, satisfying entropy condition $(E): \exists E\ \forall a>0,\ t>0,\ x\in\mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t} . \qquad (E)$$

Then $u = v$ almost everywhere in $t > 0$.

Rmk 1: we call such a solution – an entropy sol.

Rmk 2: If we had a linear operator, then the main idea of the proof could be as follows (we will adapt this idea to non-linear)

Let H be a Hilbert space.

$A: H \to H$, $\eta(A) = \{g \in H : A(g) = 0\}$ - null space

$R(A) = \{f \in H : \exists g \in H : A(g) = f\}$ - range of A

A^* is the adjacent operator:

$$\langle Ax, y\rangle = \langle x, A^* y\rangle$$

$R(A^*)$ is the orthogonal complement of $\mathcal{N}(A)$

Fact: $R(A^*) \oplus \mathcal{N}(A) = H$

The "bigger" is $R(A^*)$, the "smaller" is $\mathcal{N}(A)$.
That means that if there exist sufficiently many solutions to the adjoint equation, then the null space of A is zero $\Rightarrow$ A has a unique solution !

If $Ax = Ay$ we can choose $w: \; A^* w = x-y$:

$$\|x-y\| = \langle x-y, x-y \rangle = \langle x-y, A^* w \rangle = \langle Ax - Ay, w \rangle = 0$$

$\Rightarrow \; x = y$ (idea of Holgrem ~ 1901)

But we have a nonlinear eq!

Let us adapt this idea.

Proof of thm 2.

Let u, v be 2 solutions of $(**)$.
In order to prove that $u = v$ a.e. in $t > 0$ it suffices to show that $\forall \varphi \in C_0^1$:

$$\iint_{t>0} (u-v)\varphi = 0 .$$

What we know? Let $\psi \in C_0^1$, then

(1) $$\iint_{t\geq 0} [u \psi_t + f(u)\psi_x] \, dx\, dt + \int_{t=0} u_0 \psi \, dx = 0$$

(2) $$\iint_{t\geq 0} [v \psi_t + f(v)\psi_x] \, dx\, dt + \int_{t=0} u_0 \psi \, dx = 0$$

Subtract (1) − (2) and we get:

$$\iint_{t\geq 0} (u-v) \Big[\psi_t + \underbrace{\frac{f(u)-f(v)}{u-v}}_{=: F(x,t)} \cdot \psi_x \Big] dx\, dt = 0$$

$$\iint_{t\geq 0} (u-v) [\psi_t + F \psi_x] \, dx\, dt = 0$$

$\text{?``} \; \varphi \in C_0^1$

Now if for $\forall\ \varphi \in C_0^1$ we could solve the linear (adjoint!) equation and have a solution $\psi \in C_0^1$, we could conclude that $u = v$ a.e.

However, there is an obstruction to this approach: "velocity field" F is not smooth (not even continuous), so it is not clear why solution $\psi \in C_0^1$.

To struggle this difficulty, one can approximate u and v by smooth functions and solve corresponding linear eqs:

$$(M) \qquad \psi_t^m + F_m \psi_x^m = \varphi, \quad F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m}$$

Then
$$\iint_{t\geq 0} (u-v)\varphi = \iint_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$$

$$= \underbrace{-\iint_{t\geq 0} (u-v)\left[\psi_t^m + F\psi_x^m\right]}_{=0} + \iint_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$$

$$= \iint_{t\geq 0} (u-v)\cdot\left[F_m - F\right]\cdot \psi_x^m$$

If $F_m \to F$ locally in L_1

ψ_x^m is bounded (independently of m), then we could pass to the limit and get $=0$.

So our plan is:

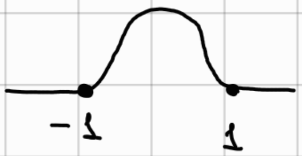
(1) approximate u, v by smooth functions u_m, v_m such that
$$\left.\begin{aligned} u_m &\to u \\ v_m &\to v \\ F_m &\to F \end{aligned}\right\} \text{ locally in } L_1$$

(2) show that for $\forall \varphi \in C_0^1$ there exists $\psi^m \in C_0^1$ – a solution of $\psi^m_t + F_m \psi^m_x = \varphi$ and it's derivative ψ^m_x is bounded (independently of m)

We will use entropy ineq. (E) HERE!

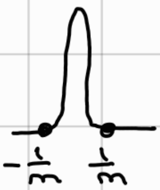
Step (1): One of the classical ideas to get a "smoother" function from any function u is to use convolution with "good kernel".

Consider $\omega(x)$ the standard "hat" function (bump)



$$\omega(x) = \begin{cases} e^{-\frac{1}{|x|^2-1}}, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

$\omega_m = c_m \, m \, \omega(mx)$ is a "hat" on the interval $[-\frac{1}{m}, \frac{1}{m}]$



Properties:

1) $\omega_m \in C^\infty(\mathbb{R})$ (exercise)
2) $\omega_m \geq 0$, $\operatorname{supp}(\omega_m) = [-\frac{1}{m}, \frac{1}{m}]$
3) $\int_{\mathbb{R}} \omega_m = 1$
4) $\omega_m \xrightarrow[m \to \infty]{} \delta(x)$

Let $u_m = u * \omega_m$ and $v_m = v * \omega_m$, where

$$(f * g)(y) = \int_{\mathbb{R}} f(x) g(y-x) \, dx \quad \text{– convolution}$$

Have in mind such a picture

$f * g$ at point y is just averaging of f in a small neighbourhood of point y.

See 3blue 1brown about convolution

Properties of $u_m = u * \omega_m$:

First studied by Kurt Otto Friedrichs (1944) and Sergey Sobolev (1938)

ⓐ $u \in L^1_{loc} \Rightarrow u_m \in C^\infty$

ⓑ $u_m \to u$ in L^1_{loc}

ⓒ $F_m \to F$ in L^1_{loc}.

Proof:

a)
$$u_m(y) = \int_{\mathbb{R}} u(x)\, \omega_m(y-x)\, dx$$

$$\frac{u_m(y+h) - u_m(y)}{h} = \int_{\mathbb{R}} u(x) \cdot \frac{\omega_m(y+h-x) - \omega_m(y-x)}{h}\, dx$$

$\downarrow$ Lebesgue theorem

$$\int_{\mathbb{R}} u(x) \cdot \frac{\partial}{\partial y} \omega_m(y-x)\, dx \quad \text{etc}$$

and $\omega_m \in C^\infty$

b)
$$u_m - u = \int_{\mathbb{R}} \omega_m(y-x)\,[u(x) - u(y)]\, dx =$$
$$= \int_{\mathbb{R}} \omega_m(z)\,[u(y+z) - u(y)]\, dz =$$
$$= \int_{-\frac{1}{m}}^{1/m} \omega_m(z)\,[u(y+z) - u(y)]\, dz$$

$$\Rightarrow \int_K |u_m - u|\, dy \le \int_K dy \int_{-\frac{1}{m}}^{\frac{1}{m}} \omega_m(z)\,[u(y+z) - u(y)]\, dz$$

K - compact set

$$\le \underbrace{\int_{-\frac{1}{m}}^{1/m} \omega_m(z)\, dz}_{=1} \cdot \underbrace{\sup_{|z| < \frac{1}{m}} \int_K [u(y+z) - u(y)]\, dy}_{\downarrow\, m \to \infty \quad 0}$$

$\Rightarrow u_m \to u$ in L^1_{loc}.

c) Write F_m as follows:

$$F_m(x,t) = \frac{f(u_m) - f(v_m)}{u_m - v_m} = \frac{1}{u_m - v_m} \int_{u_m}^{v_m} f'(s)\, ds = \int_0^1 f'(u_m \theta + v_m(1-\theta))\, d\theta$$

Analogously, $F(x,t) = \int_0^1 f'(u\Theta + v(1-\Theta))\,d\Theta$.

Let $c := \max_{|u| \le M} |f''(u)|$. Then

$$F - F_m = \int_0^1 \left[f'(u\Theta + (1-\Theta)v) - f'(u_m\Theta + (1-\Theta)v_m) \right] d\Theta =$$

$$= \int_0^1 f''(\xi) \left[\Theta(u-u_m) + (1-\Theta)(v-v_m) \right] d\Theta \text{, where}$$

ξ is between $\Theta u + (1-\Theta)v$ and $\Theta u_m + (1-\Theta)v_m$.

Due to estimates $|u|, |v|, |u_m|, |v_m| \le M$, we have $|\xi| \le M$.

Thus,

$$|F(x,t) - F_m(x,t)| \le c \int_0^1 \left[\Theta |u-u_m| + (1-\Theta)|v-v_m| \right] d\Theta \le$$

$$\le c \left(|u-u_m| + |v-v_m| \right)$$

Then for any compact set K in $\{t \ge 0\}$

$$\iint_K |F(x,t) - F_m(x,t)| \le c \underbrace{\iint_K |u-u_m|}_{\to 0} + c \cdot \underbrace{\iint_K |v-v_m|}_{\to 0} \to 0$$

$\blacksquare$

Lecture 11: Let's finish proving uniqueness of entropy sol.

Reminder: Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

  $f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f'' > 0$.

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\, dx\, dt + \int_{t=0} u_0 \varphi\, dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

Thm 2 (!) :

Let u, v be 2 solutions of $(**)$, satisfying entropy condition (E): $\exists E\ \forall a>0,\ t>0,\ x \in \mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t}. \qquad (E)$$

Then $u = v$ almost everywhere in $t>0$.

Proof:

▶ Our plan is as follows:

1. We want to show that $\forall \varphi \in C_0^1$:

$$\iint_{t>0} (u-v)\varphi = 0 \quad [\Rightarrow \quad u = v \ \text{a.e.}]$$

From $(**)$ we have $\iint_{t>0} (u-v)[\psi_t + F(x,t)\psi_x] = 0$ $\forall \psi \in C_0^1$

$$\text{for } F(x,t) = \frac{f(u(x,t)) - f(v(x,t))}{u(x,t) - v(x,t)}.$$

So if $\forall \varphi \in C_0^1\ \exists \psi \in C_0^1$ such that

$$\psi_t + F(x,t)\psi_x = \varphi \quad \text{- we would be done!}$$

Unfortunately this is not true as u, v can be discontinuous and F is not necessarily smooth.

We need to use a PDE trick - "smoothen" u, v

(2) Consider

$$u_m = u * \omega_m \in C^\infty \quad ; \quad u_m \xrightarrow{L^1} u$$
$$v_m = v * \omega_m \in C^\infty \quad ; \quad v_m \xrightarrow{L^1} v$$
$$F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m} \quad ; \quad F_m \xrightarrow{L^1} F$$

We have identity: fix $\varphi \in C_0^1$: it is enough to prove

$$\iint_{t\geq 0} (u-v)\,\varphi = \iint_{t \geq 0} (u-v)\,[F_m - F] \cdot \psi_x^m \xrightarrow[m\to\infty]{} 0$$

where ψ_m is the solution of the equation:

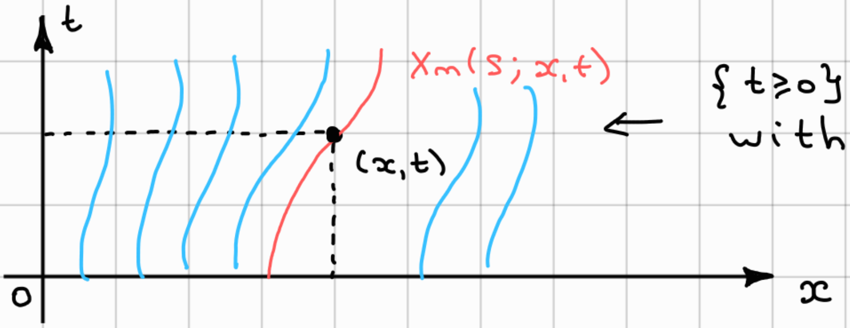
$$(M_1) \begin{cases} \psi_t^m + F_m(x,t)\,\psi_x^m = \varphi \\ \psi^m(x,T) = 0 \end{cases}$$

Here we may choose T so big such that $\varphi(x,t) = 0$ for $t \geq T$.

Notice that as F_m at least C^1, we obtain that the characteristic ODE:

$$\begin{cases} \dfrac{dx_m}{ds} = F_m(x_m, s) \\ x_m\big|_{s=t} = x \end{cases}$$

has a unique solution $x_m(s)$. It will be important for us the initial point (x,t), so we will denote such solution $x_m(s; x,t)$.



$\{t \geq 0\}$ is folliated with characteristics

<u>Lemma</u> (solution to inhomogeneous transport equation)

The solution of the problem (M_1) is given by:

$$\psi^m(x,t) = \int_T^t \varphi(x_m(s;x,t), s)\,ds.$$

Proof: This is once again Duhamel principle!

Indeed, let's check directly:

$$\psi^m_t = \underbrace{\varphi(x_m(t;x,t),t)}_{\varphi(x,t)} + \int_T^t \frac{d}{dt}\varphi(x_m(s;x,t),s)\,ds$$

$$\psi^m_x = \int_T^t \frac{d}{dx}\varphi(x_m(s;x,t),s)\,ds$$

Thus, $$\psi^m_t + F\psi^m_x = \varphi + \underbrace{\int_T^t \left[\frac{d}{dt} + F\frac{d}{dx}\right]\varphi(x_m(s;x,t),s)\,ds}_{=0}$$

Indeed, $\left[\frac{d}{dt} + F\frac{d}{dx}\right]$ is the derivative along the characteristics. But if we move the starting point (x,t) along characteristics, the function φ does not change $\Rightarrow \left[\frac{d}{dt} + F\frac{d}{dx}\right]\varphi(x_m(s;x,t),s) = 0 \quad \forall s.$

Corollary of Lemma: $\psi^m \in C_0^1(t\geq 0)$.

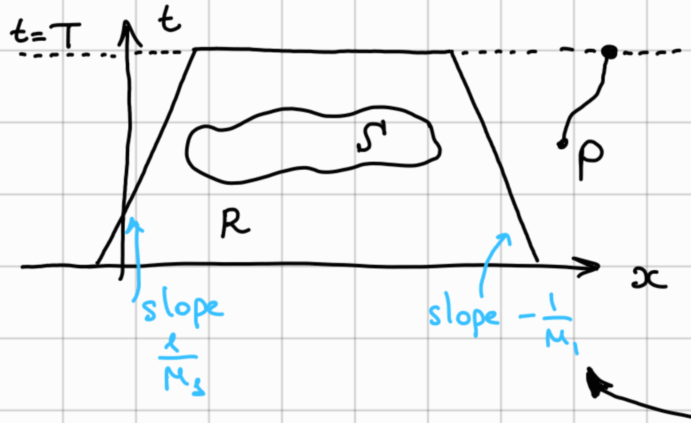
Proof:

- By Lemma $\psi^m = \int_T^t \varphi(x_m(s;t,x),s)\,ds$

  As $\varphi \in C^1(t\geq 0) \Rightarrow \psi^m \in C^1(t\geq 0)$.

  Why ψ^m has a compact support?

  Let S be support of φ (as $\varphi \in C_0^1$)



As $F_m(x,t) = \int_0^1 f'(\theta u_m + (1-\theta)v_m)\,d\theta$

$\Rightarrow |F_m| < M_1$ (as $f \in C^2$)

Consider a trapezoid R as on the figure:

(a) $S \subset R$

(b) R is bounded by four lines: $t=0$, $t=T$

and $t = -\frac{1}{M_1}x + const_1$; $t = \frac{1}{M_1}x + const_2$

Let's show that $\psi^m \equiv 0$ out of R:

1. $\psi^m = 0$ for $t \geq T$ because $\varphi \equiv 0$ there

2. Take $P = (x_1, t_1) \notin R$, $t_1 < T$.

$x_m(s; x_1, t_1) \notin R \quad \forall s \Rightarrow x_m(T; x_1, t_1) \notin R$

$\Rightarrow \varphi(x_m(s; x_1, t_1), s) = 0 \quad \forall s \Rightarrow \psi^m = 0.$ ∎

<u>Lemma</u> (boundedness of $|\psi^m_x|$)

$\exists\, C$ (independent of m):

$$|\psi^m_x| < C$$

<u>Proof</u>:

▶ The main ingredient of proof is the entropy condition : $\forall a > 0,\ t > 0$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}.$$

We see that close to $t=0$ the entropy condition spoils ($\frac{E}{t} \to \infty$ as $t \to 0$).

Let $d > 0$ be arbitrary. Then for $\forall\, t \geq d$ the function $u(x,t) - \frac{Ex}{d}$ is non-increasing in x:

$$u(x+a,t) - \frac{E(x+a)}{d} - u(x,t) + \frac{Ex}{d} \leq \frac{Ea}{t} - \frac{Ea}{d} = \underbrace{Ea\left(\frac{1}{t} - \frac{1}{d}\right)}_{\leq 0}$$

In what follows we will consider 2 cases:

(a) $t \geq d$ (b) $0 \leq t \leq d$

$\boxed{\text{Case } t \geq d}$

<u>Claim</u>: $\frac{\partial u_m}{\partial x} \leq \frac{E}{d}$; $\frac{\partial v_m}{\partial x} \leq \frac{E}{d}$

and $\exists\, K = K_d$: $\frac{\partial F_m}{\partial x} \leq K_d$

▶ Indeed, the function $\omega_n * \left(u - \frac{Ex}{d}\right) = u_m - \frac{E\, \omega_n * x}{d}$ is also non-increasing (and smooth)

$$\Rightarrow \frac{\partial}{\partial x}\left(u_m - \frac{E\, \omega_n * x}{d}\right) = \frac{\partial u_m}{\partial x} - \frac{E}{d} \leq 0$$

Analogously, $\frac{\partial v_m}{\partial x} \leq \frac{E}{\alpha}$.

$$\frac{\partial F_m}{\partial x} = \int_0^1 f''(\theta u_m + (1-\theta) v_m) \left[\theta \frac{\partial u_m}{\partial x} + (1-\theta) \frac{\partial v_m}{\partial x} \right] d\theta$$

$$\Rightarrow \frac{\partial F_m}{\partial x} \leq \int_0^1 f''(\theta u_m + (1-\theta) v_m) \left[\theta \frac{E}{\alpha} + (1-\theta) \frac{E}{\alpha} \right] d\theta$$

$$= \frac{E}{\alpha} \int_0^1 f''(\theta u_m + (1-\theta) v_m)\, d\theta$$

Therefore, $\frac{\partial F_m}{\partial x} \leq K_\alpha = \frac{E}{\alpha} \max_{|u| \leq M} f''(u)$. ∎

Let's use this to prove $\left| \frac{\partial \varphi^m}{\partial x} \right| \leq C$, $t \geq \alpha$

$$\frac{\partial \varphi^m}{\partial x} = \int_T^t \underbrace{\frac{\partial \varphi}{\partial x_m}}_{\text{is bounded}} \cdot \frac{\partial x_m}{\partial x}(s; x, t)\, ds$$

Let's examine $\frac{\partial x_m}{\partial x}$.

For convinience, denote $a_m(s) = \frac{\partial x_m}{\partial x}(s; x, t)$

Here (x,t) – some fixed point in $\{t>0\}$.

Notice $x_m(t; x, t) = x$

$$\Rightarrow a_m(t) = \frac{\partial x_m}{\partial x} = 1.$$

How $a_m(s)$ is changing with s?

$$\frac{\partial a_m}{\partial s} = \frac{\partial}{\partial s} \frac{\partial x_m}{\partial x} = \frac{\partial}{\partial x} \frac{\partial x_m}{\partial s} = \frac{\partial}{\partial x} F_m(x_m, s) =$$

$$= \frac{\partial}{\partial x} F_m(x_m(s; x, t), s) = \frac{\partial F_m}{\partial x} \cdot \frac{\partial x_m}{\partial x} =$$

$$= \frac{\partial F_m}{\partial x} \cdot a_m \Rightarrow \boxed{\frac{\partial a_m}{\partial s} = \frac{\partial F_m}{\partial x} \cdot a_m}$$

We can solve it: $a_m(s) = \exp\left(\int_t^s \frac{\partial F_m}{\partial x}(x_m(\tau), \tau)\, d\tau \right)$

Since we have $\alpha \leq t \leq s \leq T$

$$\left| \frac{\partial x_m}{\partial x} \right| = |a_m(s)| = a_m(s) \leq e^{K_\alpha (s-t)} \leq e^{K_\alpha (T-\alpha)}$$

Thus, $\left| \frac{\partial \psi^m}{\partial x} \right| \leq \int_T^t \left| \frac{\partial \varphi}{\partial x} \right| \cdot \left| \frac{\partial x_m}{\partial x} \right| ds \leq$

$$\leq (T-\alpha) \cdot C_1 \cdot e^{K_2(T-\alpha)} =: C$$

The most important is that C does not depend on m!

$\boxed{\text{Case } 0 \leq t \leq \alpha}$ Consider the total variation of ψ^m as a function of x

$$V_t(\psi^m) = \int_{\mathbb{R}} \left| \frac{\partial \psi^m}{\partial x} \right| dx \qquad \text{for each fixed } t>0.$$

As $\psi^m \in C_0^1$ and for $t \geq \alpha$ $\left| \frac{\partial \psi^m}{\partial x} \right| \leq C$ we have

$$V_t(\psi^m) \leq C_\alpha, \quad t \geq \alpha$$

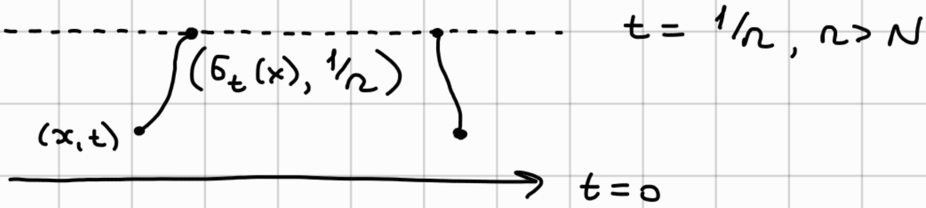
τ does not depend on m.

Rmk: let's show that $\exists N \ \forall n > N$

$$V_t(\psi^m) \leq C_{1/n} \quad \forall t : 0 < t < \frac{1}{n} < \frac{1}{N}$$

Since φ has a compact support in $\{t>0\}$, there exists N : $\varphi(x,t) = 0$ if $t < 1/N$.

Thus, $\psi^m_t + F_m \psi^m_x = 0$ if $t < 1/N$



Let $\sigma_t : \mathbb{R} \to \mathbb{R}$ - bijection that takes ψ^m at time t as initial condition and sends it to solution ψ^m at time $t = \frac{1}{n}$. As ψ^m is constant along characteristics, it is clear that

$$\sum_{k=1}^{p-1} |\psi^m(x_{k+1},t) - \psi^m(x_k,t)| = \sum_{k=1}^{p-1} \left| \psi^m\left(\sigma_t(x_{k+1}), \tfrac{1}{n}\right) - \psi^m\left(\sigma_t(x_k), \tfrac{1}{n}\right) \right| \leq$$

for any finite sequence $x_1 < x_2 < \ldots < x_p$

$$\leq V_{1/n}(\psi^m) \leq C_{1/n}.$$

Let's complete the proof of thm 2.

Fix $\varepsilon>0$ - arbitrary. Take N from Rmk above.

Choose $\alpha>0$ so small s.t. $\alpha < \frac{1}{n} \le \frac{1}{N}$ and

$$4M\, M_1\, C_{1/n}\, \alpha < \frac{\varepsilon}{2}.$$

For this α choose $\tilde{M}$ so large that

$$\iint_{t>\alpha} |u-v|\cdot|F_m - F|\cdot|\varphi^m_x| < \frac{\varepsilon}{2} \quad \text{if} \quad m \geqslant \tilde{M}$$

This can be done since $|u-v| \le 2M$, $\left|\frac{\partial \varphi^m}{\partial x}\right| \le K_\alpha$ and $F_m \to F$ in L^1_{loc}.

Then $$\left|\iint_{t\geq 0} (u-v)\varphi\right| \le \iint_{t\geq\alpha} + \iint_{t<\alpha} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

see below

Now since $\alpha < \frac{1}{n} \le \frac{1}{N}$

$$\iint_{t<\alpha} |u-v|\cdot|F_m - F|\cdot|\varphi^m_x| \le 2M\cdot 2M_1 \iint_{t<\alpha} |\varphi^m_x| = 4MM_1 \int_0^\alpha\int_{\mathbb{R}} |\varphi^m_x|$$

$$= 4MM_1 \int_0^\alpha V_t(\varphi^m)\,dt \le 4MM_1\, C_{1/n}\, \alpha < \frac{\varepsilon}{2}$$

Thus, $$\iint_{t\geq 0} (u-v)\varphi = 0 \quad \forall\, \varphi \in C^1_0 \implies u = v \text{ a.e.} \quad \blacksquare$$

Lecture 12. Riemann problem:

$$(RP) \quad \begin{cases} u_t + (f(u))_x = 0 \\ u(x,0) = \begin{cases} u_\ell, & x<0 \quad \text{- left state} \\ u_r, & x>0 \quad \text{- right state} \end{cases} \end{cases}$$

As before assume $f \in C^2$, $f'' > 0$.

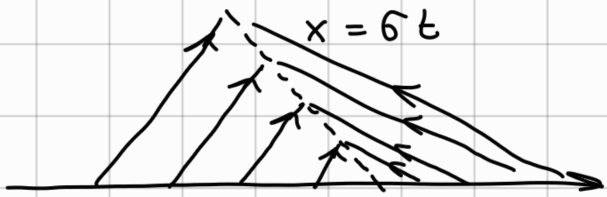
Theorem (solution to a Riemann problem):

(i) If $u_\ell > u_r$, the unique entropy solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_\ell, & \text{if } x/t < \sigma \\ u_r, & \text{if } x/t > \sigma \end{cases}$$

where

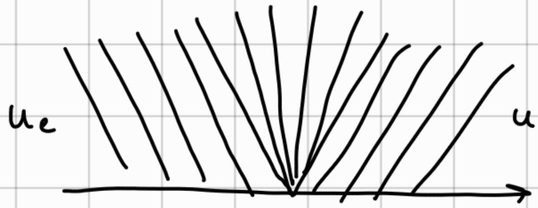
$$\sigma = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}$$

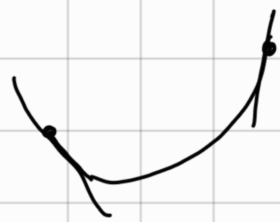


(ii) If $u_\ell < u_r$, the unique entropy solution is

$$u(x,t) = \begin{cases} u_\ell & \text{if } x/t < f'(u_\ell) \\ (f')^{-1}(x/t) & \text{if } f'(u_\ell) < x/t < f'(u_r) \\ u_r & \text{if } x/t > f'(u_r) \end{cases}$$

Such solution is called rarefaction wave





Proof:

(i) As this is shock "down" it satisfies entropy condition $\Rightarrow$ this is a unique entropy sol.

(ii) Let's look for solution of the form:

$$u(x,t) = v\left(\frac{x}{t}\right) \Rightarrow u_t + (f(u))_x = -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(v)v'\frac{1}{t}$$
$$= v'\left(\frac{x}{t}\right)\frac{1}{t}\left(f'(v) - \frac{x}{t}\right)$$

If v' never vanishes $\Rightarrow$ $f'(v) = \frac{x}{t}$ $\Rightarrow$ $v = (f')^{-1}\left(\frac{x}{t}\right)$

Also it is easy to check that v satisfies entropy cond.

Systems of conservation laws.

The most general: $u = \vec{u}(x,t) = (u_1(x,t), \dots, u_m(x,t))$

$$x \in \mathbb{R}^n, \; t \geq 0$$

$$\frac{d}{dt} \int_U u(x,t)\,dx = -\int_{\partial U} F(u)\nu\, dS = -\int_U \operatorname{div} F(u)\,dx$$

$$\Rightarrow \begin{cases} u_t + \operatorname{div} F(u) = 0, & x \in \mathbb{R}^n \\ u\big|_{t=0} = u_0 \end{cases} \qquad (*)$$

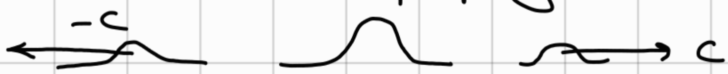
$u \in \mathbb{R}^m$ - state space. We will consider only
$F \in \mathbb{R}^m$ - flux $\qquad x \in \mathbb{R}$ $(n=1)$

Example 1: (linear) wave equation: $u_{tt} - c^2 u_{xx} = 0$

$$U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + A U_x = 0 \qquad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Eigenvalues of A: $\lambda_\pm = \pm c$

correspond to propagation modes



Example 2: (non-linear) wave equation:

$$u_{tt} - (p(u_x))_x = 0$$

$$U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + F(U)_x = 0$$

$$\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - (p(u_x))_x = 0 \end{cases} \qquad F(U) = \begin{pmatrix} -u_t \\ -p(u_x) \end{pmatrix}$$

$$U = \begin{pmatrix} v \\ w \end{pmatrix} \Rightarrow \begin{cases} v_t - w_x = 0 \\ w_t - (p(v))_x = 0 \end{cases}$$

$$F(U) = \begin{pmatrix} -w \\ -p(v) \end{pmatrix}$$

This system is called p-system (or isentropic gas dynamics)

Example 3: Euler eqs for compressible gas flow:

$$\rho_t + (\rho v)_x = 0 \quad \text{(conservation of mass)}$$
$$(\rho v)_t + (\rho v^2 + p)_x = 0 \quad \text{(conservation of momentum)}$$
$$(\rho E)_t + (\rho E v + p v)_x = 0 \quad \text{(conservation of energy)}$$

Unknowns:

- ρ – mass density
- v – velocity
- E – energy

$$p = (\rho, e)$$
$$E = \underbrace{e}_{\text{internal energy}} + \underbrace{\frac{v^2}{2}}_{\text{kinetic energy}}$$

$U = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix}$ => can be written as $U_t + F(U)_x = 0$

Weak solutions:

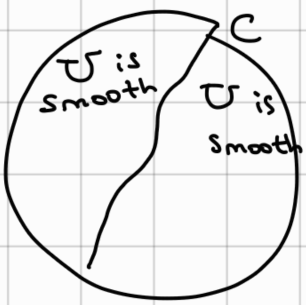
Let $v : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^m$ – smooth (C^1) with compact support, $v = (v^1, \dots, v^m)$

Do standard procedure: multiply the eq. by v and integrate by parts:

$$(**) \quad \int_0^\infty \int_{\mathbb{R}} [u \cdot v_t + F(u) v_x] \, dx\, dt + \int_{\mathbb{R}} u_0 v \, dx = 0$$

Def: we say $u \in L^\infty(\mathbb{R} \times (0, +\infty); \mathbb{R}^m)$ is a weak solution of $(*)$ provided $(**)$ holds for all v as above.

Lemma (Rankine-Hugoniot condition)



U has a jump discontinuity at C parametrized by smooth function $s(\cdot) : [0, +\infty) \to \mathbb{R}$ $(x, t) = (s(t), t)$ and let U_e be left values of U along the curve C;

U_r be right values of U along the curve C.

Then:

$$F(U_\ell) - F(U_r) = \dot{s}\,(U_\ell - U_r) \qquad \text{(RH)}$$

Rmk 1: proof is totally analogous to the scalar case – we omit it

Rmk 2: this equality (RH) is vector!

• What fluxes are reasonable?

Consider a wider class of semilinear systems

(SL) $$u_t + B(u)\,u_x = 0 \quad , \quad B: \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$$

If our solutions of $(*)$ are smooth, this system $(*)$ is equivalent to $u_t + DF \cdot u_x = 0$

$$B = DF = \begin{pmatrix} F^1_{z_1} & \dots & F^1_{z_m} \\ \vdots & \ddots & \vdots \\ F^m_{z_1} & \dots & F^m_{z_m} \end{pmatrix}_{m \times m}$$

Let's find formally the solutions in the form of a travelling wave:

$$u(x,t) = v(x - \sigma t) \Rightarrow -\sigma v' + B(v)v' = 0$$

Here v - profile, σ - velocity | Observe that this means that σ is eigenvalue of $B(u)$ and v' is an eigenvector.

If we want have some waves propagating, we should make some sort of "hyperbolicity" condition.

Def: If for each $z \in \mathbb{R}^m$ all eigenvalues of $B(z)$ are real and distinct, we call the system (SL) strictly hyperbolic.

From now on we will assume the system (SL) always strictly hyperbolic. We will write

(i) $\lambda_1(z) < \lambda_2(z) < \dots < \lambda_m(z)$, $z \in \mathbb{R}^m$

real and distinct eigenvalues of $B(z)$

(ii) $r_k(z)$ – eigenvectors of $B(z)$, $k=1\dots m$

$$B(z)\, r_k(z) = \lambda_k(z)\, r_k(z)$$

Strict hyperbolicity $\Rightarrow \operatorname{span}\{r_1(z), \dots, r_m(z)\} \equiv \mathbb{R}^m$
$\forall z \in \mathbb{R}^m$

(iii) $\ell_k(z)$ – eigenvectors of $B^T(z)$, correspond. to $\lambda_k(z)$

$$B^T(z)\, \ell_k(z) = \lambda_k(z)\, \ell_k(z)$$

or

$$\ell_k\, B(z) = \lambda_k\, \ell_k$$

Thus, we can regard r_k as right eigenvectors
ℓ_k as left eigenvectors

<u>Rmk</u>: $r_k \cdot \ell_s = 0$ if $k \neq \ell$

Indeed,

$$\lambda_k(\ell_s \cdot r_k) = \ell_s \cdot (\lambda_k r_k) = \ell_s (B \cdot r_k) = (\ell_s B) r_k = (\lambda_s \ell_s) r_k = \lambda_s \cdot \ell_s r_k$$

As $\lambda_k \neq \lambda_s \Rightarrow \ell_s \cdot r_k = 0$, $k \neq s$

Let us formulate some theorems that sound reasonable (without proof):

<u>Theorem</u> (invariance of hyperbolicity under change of coordinates)

Let u be smooth solution of (SL)
Assume $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth diffeo
Ψ it's inverse

Then: $\tilde{u} = \Phi(u)$ solves the strictly hyperbolic system: $\tilde{u}_t + \tilde{B}(\tilde{u})\, \tilde{u}_x = 0$

for $\tilde{B}(\tilde{z}) = D\Phi(\Psi(\tilde{z}))\, B(\psi(\tilde{z}))\, D\psi(\tilde{z})$

<u>Rmk</u>: weak solutions are not preserved under smooth nonlinear transformations of the equations: consider scalar eq: $u_t + (f(u))_x = 0$

$f'' > 0$ $\quad u \mapsto v = f'(u)$

$$\begin{aligned} v_t &= f''(u)\cdot u_t \\ v_x &= f''(u)\cdot u_x \end{aligned} \quad \Rightarrow \quad v_t + v\cdot v_x = 0$$

Burgers!

But this map doesn't map discontinuous solutions into themselves. Just write RH condition: the original eq: $s = \dfrac{f(u_r) - f(u_\ell)}{u_r - u_\ell}$

and for the transformed eq: $s = \dfrac{f'(u_\ell) - f'(u_r)}{u_\ell - u_r}$

<u>Theorem</u> (dependence of eigenvalues and eigenvectors on parameters)

Assume matrix function B is smooth, strictly hyperbolic. Then:

(i) the eigenvalues $\lambda_k(z)$ depend smoothly on z

(ii) we can select the right eigenvectors $r_k(z)$ and left eigenvectors $\ell_k(z)$ to depend smoothly on $z \in \mathbb{R}^m$ and satisfy the normalization:

$$|r_k(z)| = 1, \quad |\ell_k(z)| = 1$$

<u>Example 1</u> (continued): $c \neq 0 \Rightarrow$ system is strictly hyperbolic

<u>Example 2</u> (continued): $p' > 0 \Rightarrow$ system is strictly hyperbolic

$$D\begin{pmatrix} -w \\ -p(v) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -p' & 0 \end{pmatrix} \qquad \lambda_\pm = \pm\sqrt{p'}$$

Riemann problem (RP):

$$\begin{cases} u_t + (F(u))_x = 0, & u \in \mathbb{R}^m \\ u(x,0) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases} \end{cases}$$

We will call u_l, u_r left and right initial states

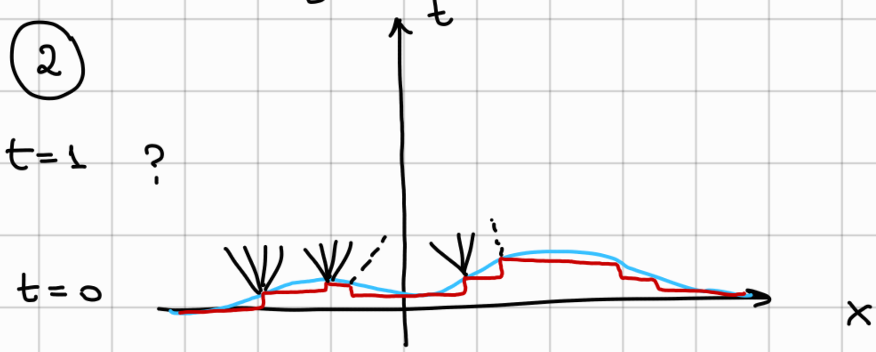
We aim at finding exact solutions to a Riemann problem. Why they are useful?

①

Often the solution to a Riemann problem appear as limiting one when $t \to +\infty$.

$u' = f(u)$ – steady states: $f(u)=0$ $\quad$ $\varepsilon u_{xx} = u_t + (f(u))_x$ – steady states: $u_t + (f(u))_x = 0$

②



One can approximate initial condition by piecewise constant initial data and solve many Riemann problems.

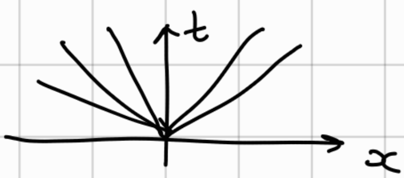
The obtained solution is some approximation

So using RP one can prove existence of solutions to Cauchy problem (with arbitrary initial data)

Rmk: notice that both equation and initial condition in RP stay the same if we consider $(x,t) \mapsto (\alpha x, \alpha t)$

Thus the solution depends only on $\frac{x}{t}$

it is constant on rays $t = kx$



Let us be engineers: to construct the general solution we need "building blocks":

→ smooth solutions ⇝ rarefaction waves

→ discontinuous solutions ⇝ shock waves

→ constant states.

§ Simple waves:

$$u(x,t) = v(w(x,t))$$
$$\left.\begin{array}{l} v : \mathbb{R} \to \mathbb{R}^m \\ w : \mathbb{R} \times (0,+\infty) \to \mathbb{R} \end{array}\right\} \text{to be found}$$

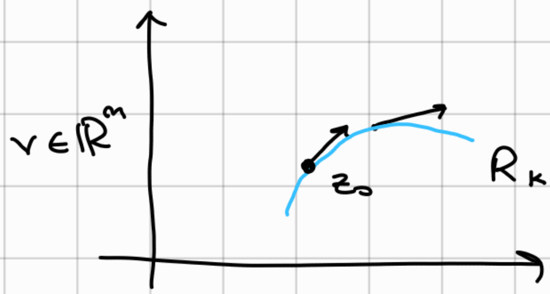
$$u_t + F(u)_x = 0 \quad \Rightarrow \quad \dot{v} \cdot w_t + \underbrace{DF \cdot \dot{v}}_{\overset{?}{=}\, \lambda_k \dot{v}} \cdot w_x = 0$$
$$\Uparrow$$

$$(SW) \begin{cases} w_t + \lambda_k(v(w))\, w_x = 0 \\ \dot{v} = r_k(v(s)) \end{cases}$$

Def: $u(x,t) = v(w(x,t))$ is called a simple wave if (SW) holds.

The main point is that we can consider first $\dot{v} = r_k(v)$, and then regard $w_t + \lambda_k w_x = 0$ as a scalar conservation law!

Def: given a fixed state $z_0 \in \mathbb{R}^m$, we define k^{th}-rarefaction curve $R_k(z_0)$ to be path in $\mathbb{R}^m$ of the solution of the ODE $\dot{v} = r_k(v)$ which passes through point z_0.



Given solution R_k we can rewrite PDE as
$$w_t + F_k(w)_x = 0$$

for $$F_k(s) = \int_0^s \lambda_k(v(t))\, dt$$

If F_k is convex, we know that the solution exists and is unique.
So this PDE will fall into general theory provided F_k is strictly convex. Let us therefore compute:

$$F_k'(s) = \lambda_k(v) \cdot \dot{v} = \lambda_k(v)$$

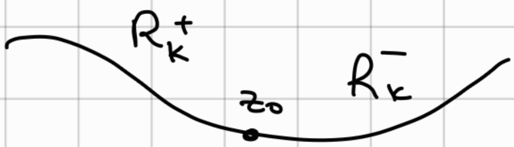
$$F_k'' = D\lambda_k \cdot \dot{v} = \underbrace{D\lambda_k(v(s)) \cdot r_k(v(s))}_{\text{this is the derivative of } \lambda_k \text{ along the k-rarefaction curve}}$$

So F_k will be convex if
$$D\lambda_k(z) \cdot r_k(z) > 0 \qquad \forall z \in \mathbb{R}^m$$
F_k – concave if
$$D\lambda_k(z) \cdot r_k(z) < 0 \qquad \forall z \in \mathbb{R}^m$$
F_k – linear if
$$D\lambda_k(z) \cdot r_k(z) \equiv 0 \qquad \forall z \in \mathbb{R}^m$$

<u>Def</u>: (i) the pair $(\lambda_k(z), r_k(z))$ is called genuinely nonlinear provided
$$D\lambda_k(z) \cdot r_k(z) \neq 0 \qquad \forall z \in \mathbb{R}^m$$

(ii) the pair $(\lambda_k(z), r_k(z))$ is called linearly degenerate provided
$$D\lambda_k(z) \cdot r_k(z) \equiv 0 \qquad \forall z \in \mathbb{R}^m$$

<u>Notation</u>: if the pair is genuinely nonlinear, write
$$R_k^+(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) > \lambda_k(z_0) \}$$
$$R_k^-(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) < \lambda_k(z_0) \}$$



$$R_k(z_0) = R_k^+(z_0) \cup \{z_0\} \cup R_k^-(z_0)$$

Lecture 13 : Reminder : we consider systems of conservation laws

$x \in \mathbb{R}$, $t > 0$, $U(x,t) = (u_1(x,t), \dots, u_m(x,t))$

$(*)$ $U_t + F(U)_x = 0$

- $U \in \mathbb{R}^m$ - state
- $F: \mathbb{R}^m \to \mathbb{R}^m$ - flux
- $DF \in M^{m \times m}$

Def: the system $(*)$ is called hyperbolic if $DF(U)$ has m real eigenvalues:

$$\lambda_1(U) \le \dots \le \lambda_m(U)$$

and the corresponding eigenvectors $r_i(U)$, $i = 1, \dots, m$, are linearly independent (form basis)

Def: the system $(*)$ is called strictly hyperbolic if $(*)$ is hyperbolic and all eigenvalues are distinct: $\lambda_1(U) < \dots < \lambda_m(U)$

In what follows we consider strictly hyperbolic systems of conservation laws.

- $U_t + B(U) U_x = 0$; eigenvalues of $B(U)$: $\lambda_1(U) < \dots < \lambda_m(U)$

$B(U)\, r_i(U) = \lambda_i(U)\, r_i(U)$, $i = 1 \dots m$

$l_i(U)\, B(U) = \lambda_i(U)\, l_i(U)$, $i = 1 \dots m$

Our goal for today: give a "constructive" proof that a Riemann problem

(RP) $U(x,0) = \begin{cases} U_l, & x < 0 \\ U_r, & x > 0 \end{cases}$ has a solution if u_l and u_r are close.

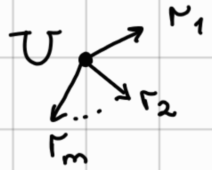
(local solution to a Riemann problem)

Our "building blocks": $i = 1 \dots m$

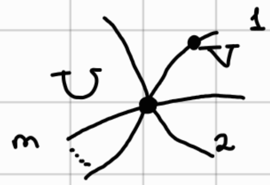
- i - rarefaction wave
- i - shock wave
- (i - contact discontinuity)
- constant states

Global picture:

$\mathbb{R}^m$: at each point U m eigenvectors

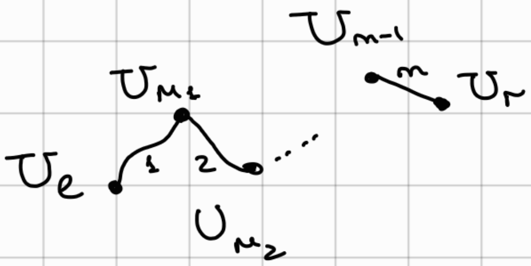


$\rightsquigarrow$ we will construct m C^1-smooth curves



such that taking point V on i-curve would mean that we have eigher smooth or discontinuous solution from U to V "corresponding to i-characteristic" $\longleftrightarrow \lambda_i$

$\rightsquigarrow$



We can construct a sequence of waves:

$$U_e \to U_{M_1} \to U_{M_2} \to \dots \to U_r$$

$\uparrow$ 1-charact. $\quad\uparrow$ 2-charac. $\quad\uparrow$ m-charac.

§ Simple waves:

$$U(x,t) = V(w(x,t))$$

$w : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ - scalar

$V : \mathbb{R} \to \mathbb{R}^m$

$$\dot V w_t + \underbrace{DF(V(w)) \cdot \dot V}_{\lambda_k \cdot V} w_x = 0 \quad , k = 1 \dots m$$

$$\begin{cases} \dot V = r_k(V(s)) \\ w_t + \underbrace{\lambda_k(V(w))} w_x = 0 \end{cases}$$

- scalar conservation law across the integral curve of the vector field induced by r_k

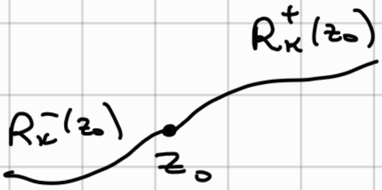
$\lambda_k(V(w))$ plays a role of the speed of propagation

$\Updownarrow$

$$w_t + (F_k(w))_x = 0 \quad \text{for} \quad F_k(s) = \int_0^s \lambda_k(V(s))\, ds$$

Def: the pair (λ_k, r_k) [or sometimes called k-characteristic family] is called genuinely nonlinear if $D\lambda_k(z) \cdot r_k(z) \neq 0 \quad \forall z \in \mathbb{R}^m$

- is called linearly degenerate if $D\lambda_k \cdot r_k \equiv 0$



$R_k^+(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) > \lambda_k(z_0) \}$

$R_k^-(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) < \lambda_k(z_0) \}$

$R_k(z) = R_k^+(z_0) \cup \{z_0\} \cup R_k^-(z_0)$

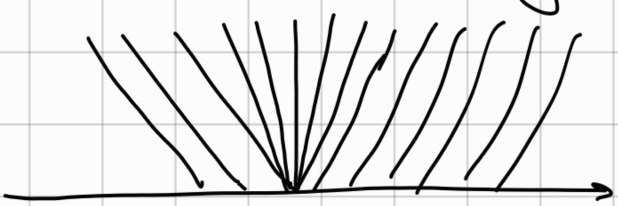
Thm (existence of k-rarefaction waves):

Suppose that for some $k = 1, \dots, m$:

(i) the pair (λ_k, r_k) is genuinely nonlinear

and (ii) $U_r \in R_k^+(z_0)$.

Then there exists a continuous integral solution U of a Riemann problem (RP), which is a k-simple wave constant along lines through origin



Rmk: if $U_r \in R_k^-$, then such a cont. sol. doesn't exist!

Proof:

1. Take $w_l, w_r \in \mathbb{R}$: $U_l = V(w_l)$; $U_r = V(w_r)$

   Suppose $w_l < w_r$.

2. Consider a scalar Riemann problem consisting of PDE

$$\begin{cases} w_t + (F_k(w))_x = 0 \\ w(x,0) = \begin{cases} w_l, & x<0 \\ w_r, & x>0 \end{cases} \end{cases}$$

$F_k' = \lambda_k(V(s))$, $F_k'' = D\lambda_k(V(s)) \cdot r_k(V(s)) \underset{(i)}{\neq} 0$

(ii) $\Rightarrow$ $\lambda_k(U_r) > \lambda_k(U_l)$

$\Rightarrow$ $F_k'(w_r) > F_k'(w_l)$ $\Rightarrow$ F - strictly convex

$\Rightarrow$ this scalar conservation law admits a continuous solution - a rarefaction wave

$$w(x,t) = \begin{cases} w_l & \text{if } \frac{x}{t} < F_k'(w_l) \\ (F_k')^{-1}\left(\frac{x}{t}\right) & \text{if } F_k'(w_l) < \frac{x}{t} < F_k'(w_r) \\ w_r & \text{if } F_k'(w_r) < \frac{x}{t} \end{cases}$$

Thus $U(x,t) = V(w(x,t))$ solves PDE. The case $w_l > w_r$ is treated similarly (F_k is concave) ∎

Shock waves: by RH condition, $\sigma \in \mathbb{R}$ - a shock wave speed

$$F(U_l) - F(U_r) = \sigma(U_l - U_r)$$

Def: for a given (fixed) state $U_0 \in \mathbb{R}^m$ we define a shock set (Hugoniot locus)

$$S(U_0) = \{U \in \mathbb{R}^m : \exists \sigma \in \mathbb{R} : F(U) - F(U_0) = \sigma(U - U_0)\}$$

$$\sigma = \sigma(U, U_0)$$

That is this is a set of all states to which there possibly exist a shock wave (with some speed) from U_0.

Thm (structure of shock set)
Fix $U_0 \in \mathbb{R}^m$. In some neighborhood of U_0 $S(U_0)$ consists of the union of m smooth curves $S_k(U_0)$, $k = 1 \dots m$, with the following properties:

(i) The curve $S_k(U_0)$ passes through U_0 with tangent $r_k(U_0)$

(ii) $\lim\limits_{U \to U_0} \sigma(U, U_0) = \lambda_k(z_0)$

(iii) $\sigma(U, U_0) = \dfrac{\lambda_k(U) + \lambda_k(U_0)}{2} + O(|U - U_0|^2)$

as $U \to U_0$ with $U \in S_k(U_0)$.

Proof:

$$F(U) - F(U_0) = B(U)(U - U_0), \text{ where}$$

$$B(U) = \int_0^1 DF(U_0(1-t) + Ut)\,dt, \quad U \in \mathbb{R}^m$$

— "averaged" Jacobi matrix DF

$U \in S(U_0)$ iff $\quad (B(U) - \sigma I)(U - U_0) = 0 \quad (1)$

for some scalar $\sigma = \sigma(U, U_0)$.

$$B(U_0) = DF(U_0)$$

Strict hyperbolicity $\Rightarrow$ $\det(\lambda I - B(U_0))$ has m distinct real roots

$\Rightarrow$ $\det(\lambda I - B(U))$ has m distinct real roots if U is close to U_0.

More over, $\hat{\lambda}_1(U) < \dots < \hat{\lambda}_m(U)$ are smooth functions of U and $\hat{r}_k(U)$, $\hat{\ell}_k(U)$ unit vectors:

$$\hat{\lambda}_k(U_0) = \lambda_k(U_0)$$
$$\hat{r}_k(U_0) = r_k(U_0)$$
$$\hat{\ell}_k(U_0) = \ell_k(U_0)$$

and

$$B(U)\hat{r}_k(U) = \hat{\lambda}_k(U)\hat{r}_k(U) \qquad k = 1 \dots m$$
$$\hat{\ell}_k(U)B(U) = \hat{\lambda}_k(U)\hat{\ell}_k(U)$$

Note that both $\{\hat{r}_k\}$ and $\{\hat{\ell}_k\}$ are bases of $\mathbb{R}^m$ and $\hat{r}_k \cdot \hat{\ell}_n = 0$, $n \neq k$.

Eq. (1) will hold provided $\sigma = \hat{\lambda}_k$ for some k and $U - U_0$ is parallel to $\hat{r}_k$. This is equivalent to:

$$\hat{\ell}_\ell(U) \cdot (U - U_0) = 0, \quad \ell \neq k$$

These are $(m-1)$ equations to m components of U, so we can use Implicit Function Theorem to solve it.

Define $\Phi_k : \mathbb{R}^m \to \mathbb{R}^{m-1}$

$$\Phi_k(U) = (\dots, \hat{\ell}_{k-1}(U)(U - U_0), \hat{\ell}_{k+1}(U)(U - U_0), \dots)$$

$$\Phi_k(U_0) = 0 \quad \text{and} \quad D\Phi_k(U_0) = \begin{pmatrix} \ell_1(U_0) \\ \vdots \\ \ell_{k-1}(U_0) \\ \ell_{k+1}(U_0) \\ \dots \end{pmatrix}$$

Since $\{e_i\}$ form a basis, we have

$$\text{rank } D\Phi_k(U_0) = m-1$$

Hence, $\exists$ a smooth curve $\varphi_k : \mathbb{R} \to \mathbb{R}^m$ such that $\varphi_k(0) = U_0$ and

$$\Phi_k(\varphi_k(t)) = 0 \quad \forall\, t \text{ close to } 0.$$

The path of curve φ_k define $S_k(U_0)$.
We may choose parametrization:

$$|\dot{\varphi}_k(t)| = 1$$

Thus we have found m smooth curves $S_k(U_0)$. Let us now properties (i)-(iii)

Property (i):

$$\varphi_k(t) - U_0 = \mu(t) \cdot \hat{r}_k(\varphi_k(t))$$

where μ is a smooth function satisfying $\mu(0) = 0$, $\dot{\mu}(0) = 1$

Thus, $\dot{\varphi}_k(0) = \hat{r}_k(U_0) = r_k(U_0)$ at U_0
Hence, the curve $S_k(U_0)$ has tangent $r_k(U_0)$

Property (ii): According to what we have proved, there exists a smooth function $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$: $\forall\, t$ close to 0

$$F(\varphi_k(t)) - F(U_0) = \sigma(\varphi_k(t), U_0)\,(\varphi_k(t) - U_0)$$

Thus,

$$DF(U_0) \cdot \dot{\varphi}_k(0) = \sigma(U_0, U_0)\,\dot{\varphi}_k(0)$$

$$\Rightarrow \sigma(U_0, U_0) = \lambda_k(U_0)$$

Property (iii): for simplicity write $\sigma(t) = \sigma(\varphi_k(t), U_0)$

$$F(\varphi_k(t)) - F(U_0) = \sigma(t)\,(\varphi_k(t) - U_0).$$

Differenciate twice wrt t:

$$\frac{d}{dt}: \quad DF(\varphi_k(t)) \cdot \dot{\varphi}_k(t) = \dot{\sigma}\,(\varphi_k(t) - U_0) + \sigma \cdot \dot{\varphi}_k$$

$$\frac{d^2}{dt^2}: \quad \left(D^2F(\varphi_k(t)) \cdot \dot{\varphi}_k\right) \dot{\varphi}_k + DF(\varphi_k(t)) \cdot \ddot{\varphi}_k =$$
$$= \ddot{\sigma}\,(\varphi_k - U_0) + 2\,\dot{\sigma} \cdot \dot{\varphi}_k + \sigma\,\ddot{\varphi}_k$$

Evaluate this expression at $t=0$ $\left(\begin{array}{l} \varphi_k(0) = U_0 \\ \dot{\varphi}_k(0) = r_k(U_0) \end{array}\right)$

(2) $$\left(D^2F(U_0)\, r_k(U_0) - 2\dot{\sigma}\, I\right) r_k(U_0) = \left(\lambda_k(U_0) - DF(U_0)\right) \cdot \ddot{\varphi}_k$$

Let $\psi_k(t) = V(t)$ be a unit speed parametrization of the rarefaction curve $R_k(U_0)$ near U_0.

Then $\psi_k(0) = U_0$, $\dot{\psi}_k(t) = r_k(\psi_k(t))$

Thus, $DF(\psi_k(t))\, r_k(t) = \lambda_k(t)\, r_k(t)$

Differenciate this wrt t and evaluate at $t=0$

(3) $$\left(D^2F(U_0)\, r_k(U_0) - \dot{\lambda}_k(0)\, I\right) r_k(U_0) = -(DF + \lambda_k I)\, \dot{r}_k$$

Subtract (3) from (2) and obtain:

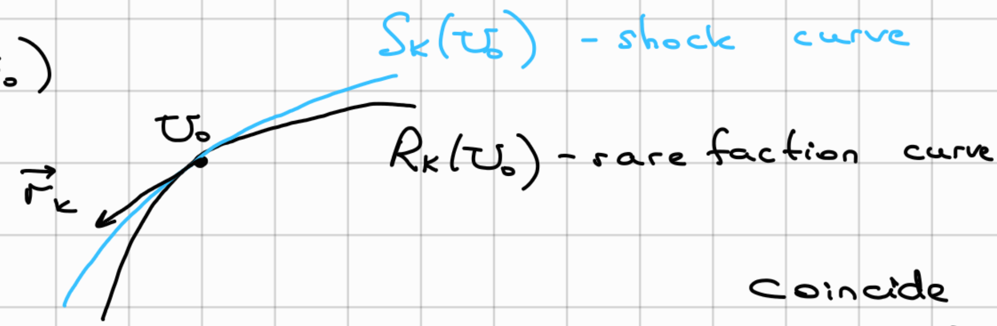
$$\left(\dot{\lambda}_k(0) - 2\dot{\sigma}\right) r_k(U_0) = \left(DF - \lambda_k I\right)\left(\dot{r}_k - \ddot{\varphi}_k\right)$$

Take dot product with $\ell_k(U_0)$, we obtain

$$\dot{\lambda}_k(0) = 2\,\dot{\sigma}(0) \Rightarrow 2\,\sigma(t) = \lambda_k(U_0) + \lambda_k(U) + O(t^2)$$

$\blacksquare$

So we have $S_k(U_0)$ and $R_k(U_0)$ agree at least to first order at U_0.



In the linearly degenerate case these curves coincide

Thm (linear degeneracy).
Suppose for some $k = 1 \dots m$ the pair (λ_k, r_k) is linearly degenerate. Then for each $U_0 \in \mathbb{R}^m$:

(i) $R_k(U_0) = S_k(U_0)$

(ii) $\sigma(U, U_0) = \lambda_k(U) = \lambda_k(U_0) \quad \forall U \in S_k(U_0)$

Proof:

Let $V = V(s)$ solve ODE
$$\begin{cases} \dot{V}(s) = r_k(V(s)) \\ V(0) = U_0 \end{cases}$$

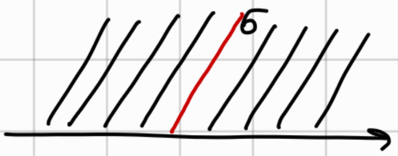
Then as $D\lambda_k \cdot r_k \equiv 0$, the mapping $s \mapsto \lambda_k(V(s))$ is constant.

So
$$F(V(s)) - F(U_0) = \int_0^s DF(V(t)) \cdot \dot{V}(t)\,dt =$$
$$= \int_0^s DF(V(t)) \cdot r_k(V(t))\,dt = \int_0^s \lambda_k(V(t))\, r_k(V(t))\,dt$$
$$= \lambda_k(U_0) \int_0^s \dot{V}(t)\,dt = \lambda_k(U_0)\big(V(t) - U_0\big)$$
■

Contact discontinuities: Let (λ_k, r_k) be linearly degenerate

$U_\ell \in \mathbb{R}^m$, $U_r \in S_k(U_\ell)$

Then $U(x,t) = \begin{cases} U_\ell, & x < \sigma t \\ U_r, & x > \sigma t \end{cases}$ $\quad \sigma = \sigma(U_\ell, U_r) = \lambda_k(U_\ell)$



k-contact discontinuity

Shock waves: Let (λ_k, r_k) be genuinely nonlinear $U_\ell \in \mathbb{R}^m$, $U_r \in S_k(U_\ell)$

Consider $U(x,t) = \begin{cases} U_\ell, & x < \sigma t \\ U_r, & x > \sigma t \end{cases}$ for $\sigma = \sigma(U_\ell, U_r)$

There are 2 essentialy different cases:

case I: $\lambda_k(U_r) < \lambda_k(U_\ell)$

case II: $\lambda_k(U_\ell) < \lambda_k(U_r)$

In view of thm of structure of shock curve, we have: case I: $\lambda_k(U_r) < \sigma(U_\ell, U_r) < \lambda_k(U_\ell)$

$\lambda_k(U_\ell) < \sigma(U_\ell, U_r) < \lambda_k(U_r)$

provided that U_r is close to U_ℓ

Def: assume the pair (λ_k, r_k) is genuinely nonlinear at U_ℓ. We say that the pair (U_ℓ, U_r) is admissible provided:

ⓐ $U_r \in S_k(U_\ell)$

ⓑ $\lambda_k(U_r) < \sigma(U_\ell, U_r) < \lambda_k(U_\ell)$

We refer to this condition as Lax entropy condition.

Def: If (U_ℓ, U_r) is admissible, we call our solution U defined as above a k-shock wave.

Def: Let $S_k^+(U_0) = \{ U \in S_k(U_0) : \lambda_k(U_0) < \sigma(U, U_0) < \lambda_k(U) \}$

$S_k^-(U_0) = \{ U \in S_k(U_0) : \lambda_k(U_0) > \sigma(U, U_0) > \lambda_k(U) \}$

Then $S_k(U_0) = S_k^+(U_0) \cup \{z_0\} \cup S_k^-(U_0)$

Note that the pair (U_ℓ, U_r) is adm. iff $U_r \in S_k^-(U_\ell)$

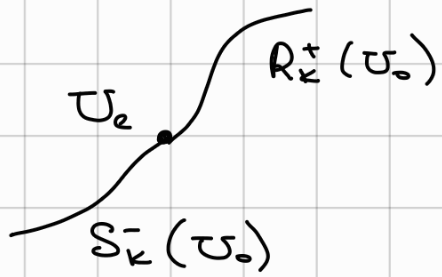
Now let us glue everything together.

<u>Def</u>: (i) if pair (λ_k, r_k) is genuinely nonlinear, write $T_k(U_0) = R_k^+(U_0) \cup \{U_0\} \cup S_k^-(U_0)$

(ii) if pair (λ_k, r_k) is linearly degenerate, write $T_k(U_0) = R_k(U_0) = S_k(U_0)$

<u>Rmk</u>: the curve $T_k(U_0)$ is C^1

So if $U_r \in T_k(U_\ell)$, then there exists a solution to a Riemann problem (being or k-rarefaction wave or k-shock wave or k-contact discontinuity)



Finally, we want to prove theorem:

<u>Thm</u> (local solution of Riemann problem)

Assume that for each $k = 1 \dots m$ the pair (λ_k, r_k) is either genuinely nonlinear or linearly degenerate. Suppose we have fixed U_ℓ. Then for each right state U_r sufficiently close to U_ℓ there exists an integral solution U of (RP) which is constant on lines through the origin.

<u>Proof</u>:

Again Implicit Function Theorem (Next time)

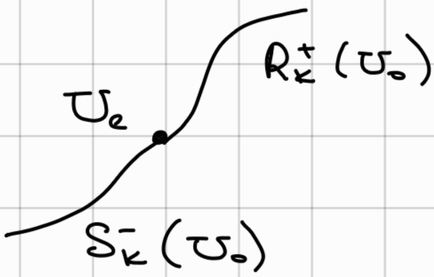
Now let us glue everything together.

<u>Def</u>: (i) if pair (λ_k, r_k) is genuinely nonlinear, write $T_k(U_0) = R_k^+(U_0) \cup \{U_0\} \cup S_k^-(U_0)$

(ii) if pair (λ_k, r_k) is linearly degenerate, write $T_k(U_0) = R_k(U_0) = S_k(U_0)$

<u>Rmk</u>: the curve $T_k(U_0)$ is C^1

So if $U_r \in T_k(U_\ell)$, then there exists a solution to a Riemann problem (being or k-rarefaction wave or k-shock wave or k-contact discontinuity)



Finally, we want to prove theorem:

<u>Thm</u> (local solution of Riemann problem)

Assume that for each $k = 1 \ldots m$ the pair (λ_k, r_k) is either genuinely nonlinear or linearly degenerate. Suppose we have fixed U_ℓ. Then for each right state U_r sufficiently close to U_ℓ there exists an integral solution U of (RP) which is constant on lines through the origin.

<u>Proof</u>:

▶ Again Implicit Function Theorem: $\Phi : \mathbb{R}^m \to \mathbb{R}^m$

First, for each family of curves T_k, $k = 1..m$, choose the nonsingular parameter τ_k to measure arc length: $\forall\, U, \tilde{U} \in \mathbb{R}^m$ with

$\tilde{U} \in T_k(U)$ we have

$\tau_k(\tilde{U}) - \tau_k(U) =$ (signed) distance from $\tilde{U}$ to U along the curve $T_k(z)$

We take "+" if $\tilde{U} \in R_k^+(U)$ and

"−" if $\tilde{U} \in S_k^-(U)$.

Second, given $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ with $|t|$ small we define $\Phi(t) = U$ as follows.

- $\Phi(0) = U_0$
- Then choose states $U_1, \dots, U_m$:

$$U_1 \in T_1(U_0), \quad \tau_1(U_1) - \tau_1(U_0) = t_1$$
$$U_2 \in T_2(U_1), \quad \tau_2(U_2) - \tau_2(U_1) = t_2$$
$$\dots\dots \qquad \dots\dots$$
$$U_m \in T_m(U_{m-1}), \quad \tau_m(U_m) - \tau_m(U_{m-1}) = t_m$$

Now write $\Phi(t) = z_m$.

- $\Phi \in C^1$
- $\Phi(0) = z_0$
- $D\Phi(0)$ is nonsingular

$$\Phi(0, \dots, t_k, \dots, 0) - \Phi(0, \dots, 0) = t_k r_k(U_0) + o(t_k), \quad t_k \to 0$$

Thus,
$$\frac{\partial \Phi}{\partial t_k}(0) = r_k(U_0) \quad \text{and so}$$

$$D\Phi(0) = (r_1(U_0), \dots, r_m(U_0))_{m \times m}$$

Since $\{r_i\}$ is a basis, $D\Phi(0)$ is nonsingular.

Thus, by the inverse function theorem

$\forall U_r$ sufficiently close to U_e $\exists!\ t = (t_1, \dots t_m)$

$\Phi(t) = U_r$.

So we get a sequence: $U_e \to U_1 \to U_2 \to \dots \to U_m$.

Recall that if U_{k-1} and U_k are joined by k-rarefaction wave, this wave is:

$$\begin{cases} U_{k-1} & \text{if } \frac{x}{t} < \lambda_k(U_{k-1}) \\ (f_k')^{-1}\left(\frac{x}{t}\right) & \text{if } \lambda_k(U_{k-1}) < \frac{x}{t} < \lambda_k(U_k) \\ U_k & \text{if } \lambda_k(U_k) < \frac{x}{t} \end{cases}$$

Moreover, if U_{k-1}, U_k are joined by k-shock, it has the form:

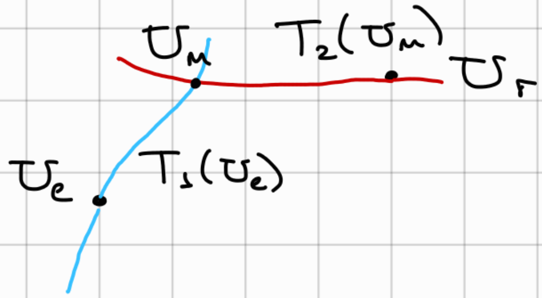
$$\begin{cases} U_{k-1} & \text{if } \frac{x}{t} < \sigma(U_k, U_{k-1}) \\ U_k & \text{if } \frac{x}{t} > \sigma(U_k, U_{k-1}) \end{cases}$$

In both cases the waves are constant outside the regions $\lambda_k(U_0) - \varepsilon < \frac{x}{t} < \lambda_k(U_0) + \varepsilon$ for small $\varepsilon > 0$ provided U_k, U_{k-1} are close enough.
This is true for $k = 1, \dots, m$.
Since $\lambda_1(U_0) < \dots < \lambda_m(U_0)$, we see that rarefactions, shock or contact discontinuities connecting

$$\begin{aligned} &U_0 \text{ to } U_1 \rightsquigarrow \approx \lambda_1(U_0) \\ &U_1 \text{ to } U_2 \rightsquigarrow \approx \lambda_2(U_0) \\ &\dots\dots \qquad\qquad \dots \\ &U_{m-1} \text{ to } U_m \rightsquigarrow \approx \lambda_m(U_0) \end{aligned}$$

do not intersect.
Thus, we have constructed a solution. ∎

$\boxed{m=2}$



Lecture 14: Solution to exercise 3 from HW 3.

$$S_t + f(S)_x = 0 \qquad f(S) - \text{S-shaped}$$

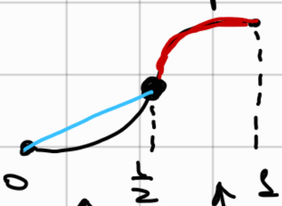
Buckley-Leverett eq
flow in porous media

- $f(0) = 0$, $f(1) = 1$
- $f' > 0$
- $\exists\, s = \frac{1}{2}$: $f''(s) > 0$, $s < \frac{1}{2}$
  $f''(\frac{1}{2}) = 0$
  $f''(s) > 0$, $s > \frac{1}{2}$.

$$S_t + \underbrace{f'(s)}_{\lambda(s)} S_x = 0$$

$D\lambda = f''(s)$ is 0 at point $s = \frac{1}{2}$
$\Downarrow$
not genuinely nonlinear
not linearly degenerate

How does the solution look like?
The naive idea is that the solution consists of 2 parts (say, we are trying to solve the Riemann problem $\overline{1}\,\underline{\quad 0}$):

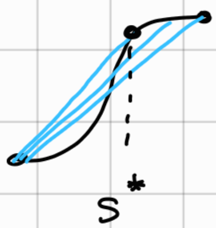


but this is a strange solution

?!

slope = shock speed
tangents = speed of S for a rarefaction wave

To avoid such situation, we see that



only $s \geq s^*$ are valid for left state of a shock. Here u^* is the abscissa of the tangent line from $u=0$ to graph of f

Then we get too many solutions: ....

On the other hand, may be not all the shocks satisfy additional entropy condition?

Let's consider the vanishing viscosity criterion

$$S_t + (f(S))_x = \varepsilon S_{xx}$$

We seek for solutions of the form

$$S = S\left(\frac{x - vt}{\varepsilon}\right) = S(\xi)$$

$$\Rightarrow \quad -v\, S' + (f(S))' = S''$$

Integrate from $\xi = -\infty$ till ξ :

$$-vS + f(S) = S'$$

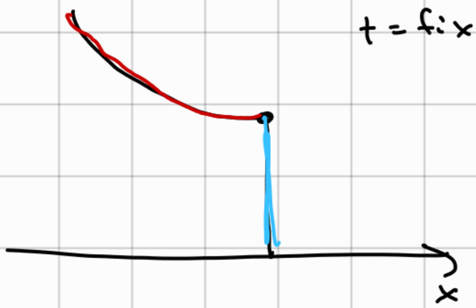
$$\begin{cases} S' = f(S) - vS \\ S(-\infty) = 0 \\ S(+\infty) = S_R \end{cases}$$

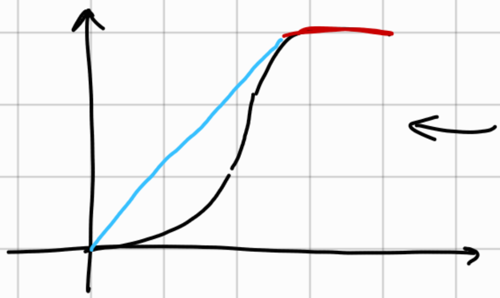
Necessary condition : $f(S) - vS < 0$

$f(S) < vS$.

This is valid only for points $S \le S^*$.

Thus, the only admissible shock wave is

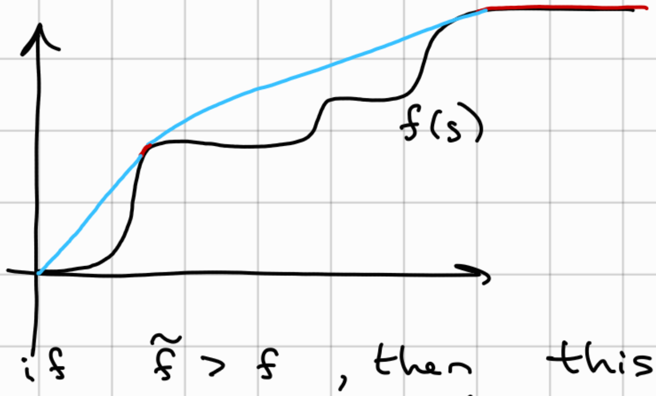




← convex hull of graph f

In general there is the following algorithm of constructing a solution to a Riemann problem

$$S(x,0) = \begin{cases} S_L, & x < 0 \\ S_R, & x > 0 \end{cases} \qquad S_L > S_R$$

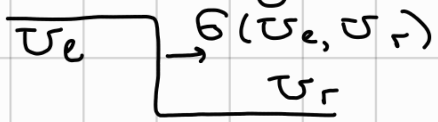


Take convex hull of $f(S)$: say $\tilde{f}(S)$:

- if $\tilde{f}(S) = f(S)$, then this S is moving with $f'(S)$ (as a part of rarefaction wave)
- if $\tilde{f} > f$, then this corresponds to a shock wave

(*) $U_t + F(U)_x = 0$, $U \in \mathbb{R}^m$

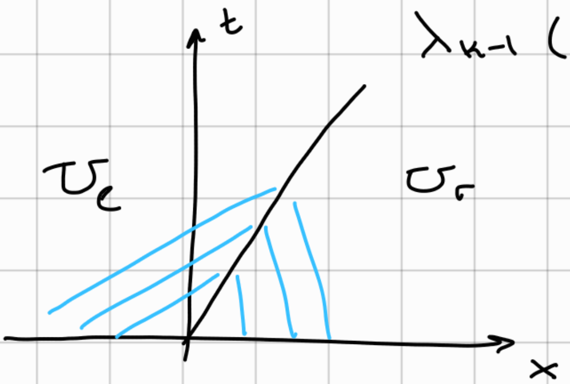
Entropy criteria for weak solutions for systems of conservation laws.



$$U(x,t) = \begin{cases} U_l, & x < \sigma t \\ U_r, & x > \sigma t \end{cases}$$

① **Lax**: $\exists k = 1..m$:

$$\lambda_k(U_r) < \sigma(U_l, U_r) < \lambda_k(U_l)$$
$$\lambda_{k-1}(U_l) < \sigma(U_l, U_r) < \lambda_{k+1}(U_r)$$



There is only one index k such that the shock speed σ is intermediate to the characteristic speed λ_k on both sides of the shock.

There exists an "empirical" explanation that the total amount of characteristics that "come" to shock should be $(n+1)$.

] $u_t + a u_x = 0$ in $x > 0$, $t > 0$:

we need to define $u(0,t)$

we do not need to define $u(0,t)$

] $U_t + A U_x = 0$ linear system

Let $P^{-1} A P = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$

Then for $V = P^{-1} U$ the system $v^i_t + \lambda_i v^i_x = 0$ decouples into m equations.

If k: $\lambda_i < 0$, $i \le k$

$\lambda_i > 0$, $i = k+1, \dots$

Thus we need $(n-k)$ conditions on the components of v (and u) on the bound. $x = 0$

] More generally, if we have a boundary that moves with speed s ($s=0$ was for quarter plane problem) and if

$$\lambda_1 < \dots < \lambda_k < s < \lambda_{k+1} < \dots < \lambda_m$$

we must give $(n-k)$ conditions on u in order to be able to specify the solution in the region $x-st>0$, $t>0$.

Now extend this to a discontinuity of a hyperbolic system: let $\lambda_1(u) < \lambda_2(u) < \dots < \lambda_m(u)$ be eigenvalues of $DF(u)$. Suppose,

$$\lambda_k(U_r) < s < \lambda_{k+1}(U_r)$$

on the right bound. of disc

Then we should specify $(n-k)$ conditions

Similarly, if we assume

$$\lambda_j(U_\ell) < s < \lambda_{j+1}(U_\ell)$$

then we must specify j conditions on the left boundary of disc.

We have $(n-1)$ jump conditions from RH:

$(n-k)+j = n-1$ or $j = k-1$. That are exactly the Lax conditions above.

② Vanishing viscosity (limit of traveling waves)

$$u^\varepsilon(x,t) = v\left(\frac{x-\sigma t}{\varepsilon}\right), \quad v: \mathbb{R} \to \mathbb{R}^m$$

Then v must solve ODE:

$$\begin{cases} \ddot{v} = -\sigma \dot{v} + (F(v))^{\cdot} \\ v(-\infty) = U_\ell, \quad v(+\infty) = U_r, \quad \lim_{\xi \to \pm\infty} \dot{v} = 0 \end{cases}$$

Integrating we get:

$$\dot{v} = F(v) - F(U_\ell) - \sigma(v - U_\ell)$$

Now the sysmem is m-dimensional and in general more difficult

Thm (existence of traveling waves for genuinely nonlinear systems)

Assume (λ_k, r_k) is genuinely nonlinear for $k = 1 \dots m$. Let U_r be sufficiently close to U_ℓ. Then there exists a travelling wave solution connecting U_ℓ and U_r iff $U_r \in S_k^-(U_\ell)$ for some k

(without proof)

③ Liu criterion (internal stability of a shock)

Let $U_r \in S_k(U_\ell)$ for some $k = 1 \dots m$

and $\sigma(z, U_\ell) > \sigma(U_r, U_\ell) > \sigma(U_r, z)$

for each z lying on the curve $S_k(U_\ell)$ between U_r and U_ℓ.

④ Entropy / Entropy-flux pair

Def: we say two smooth functions $\Phi, \Psi : \mathbb{R}^m \to \mathbb{R}$ comprise an entropy / entropy-flux pair for the conservation law $U_t + F(U)_x = 0$

provided: (a) Φ is convex

(b) $D\Phi(z) \cdot DF(z) = D\Psi(z), \ z \in \mathbb{R}^m$

Rmk: if solution of $U_t + F(U)_x = 0$ is smooth, then $$D\Phi \cdot U_t + \underbrace{D\Phi \cdot DF}_{D\Psi} \cdot U_x = 0$$

$$\Rightarrow \quad \Phi(U)_t + \Psi(U)_x = 0$$ – this is just an additional conservation law!

For shocks we do not have this equiality, but instead we could replace $\Phi_t + \Psi_x = 0$ with inequality: $\Phi(U)_t + \Psi(U)_x \leq 0$, $\begin{smallmatrix} t>0 \\ x\in\mathbb{R} \end{smallmatrix}$

In applications, Φ some times is negative of physical entropy and Ψ is entropy flux (this explains the terminology)

The rigorous understanding of the inequality in weak sense: $\forall\ \varphi \in C_c^\infty(\mathbb{R}\times(0,+\infty))$, $\varphi \geq 0$:

$$(EEF) \quad \int_0^\infty \int_{\mathbb{R}} \left(\Phi(U)\varphi_t + \Psi(U)\varphi_x \right) dx\,dt \geq 0$$

<u>Def</u>: we call $U \in \mathbb{R}^m$ an entropy solution of $(*)$ provided U is a weak solution of $(*)$ and satisfies inequalities (EEF) for every entropy / entropy-flux pair (Φ, Ψ)

Why such inequality? This can be easily seen if we think of u as a limit of U^ε – solution of vanishing viscosity method:

$$U^\varepsilon_t + F(U^\varepsilon)_x = \varepsilon\, U^\varepsilon_{xx} \qquad \Big|\cdot D\Phi(U^\varepsilon)$$

$$\Phi(U^\varepsilon)_t + \Psi(U^\varepsilon)_x = \varepsilon\, D\Phi(U^\varepsilon)\, U^\varepsilon_{xx}$$

$$D\Phi(U^\varepsilon)\, U^\varepsilon_{xx} = \left(\Phi(U^\varepsilon)\right)_{xx} - \left(D^2\Phi(U^\varepsilon)\, U^\varepsilon_x\right) U^\varepsilon_x$$

Φ - convex $\Rightarrow$ $\left(D^2\Phi(U^\varepsilon)\cdot U^\varepsilon_x\right) U^\varepsilon_x \geq 0$

$$\Phi(U^\varepsilon)_t + \Psi(U^\varepsilon)_x \leq \varepsilon\left(\Phi(U^\varepsilon)\right)_{xx} \quad \forall \varepsilon$$

$$\underline{\leq 0}$$

When we can find an entropy?

<u>Example</u>: $\boxed{m=1}$ scalar conservation law
for any convex Φ we can find
a flux function:

$$\Psi(z) = \int_{z_0}^{z} \Phi'(w)\, F'(w)\, dw \quad , \; z \in \mathbb{R}$$

With this notion of entropy solution one can prove that there exists at most 1 weak solution of a scalar conservation law

$\boxed{m=2}$ p-system:
$$(\Phi_{z_1}, \Phi_{z_2}) \begin{pmatrix} 0 & -1 \\ p'(z_1) & 0 \end{pmatrix} = \begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix}$$

$$\begin{cases} p'(z_1)\, \Phi_{z_2} = \Psi_{z_1} \\ -\Phi_{z_1} = \Psi_{z_2} \end{cases}$$

$$\Rightarrow \begin{cases} \Phi = \dfrac{z_2^2}{2} - \displaystyle\int_0^{z_1} p(w)\, dw \\ \Psi = p(z_1)\, z_2 \end{cases}$$

– convex as $p' < 0$

check!!!