Lecture 2

Wave equation

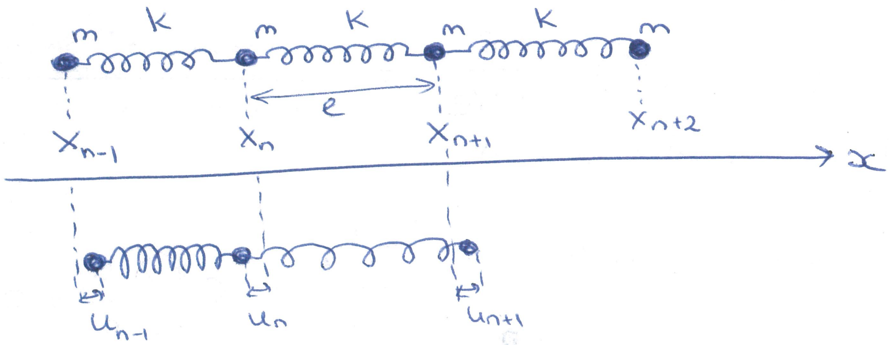
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

c - wave speed

Plan:

1. Derivation
2. D'Alembert exact solution
3. Well-posedness
4. Inhomogeneous wave equation (exercise)
5. Mixed initial-boundary value problem
   - 5a) $\exists$ and !
   - 5b) Exact solution by spectral method

1. Derivation 1: (from physics)



- Position at rest: $x_n = nl$, $n \in \mathbb{Z}$
- k - spring constant (measure of the spring's stiffness)

$(\dots, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \dots)$ - vector of horizontal displacements

Second Newton's law for the n-th mass:

$$\longrightarrow m\ddot{u}_n = F_n^- + F_n^+$$

Hooke's law: $F_n^- = -k(u_n - u_{n-1})$; $F_n^+ = k(u_{n+1} - u_n)$ where k is the elastic constant of each spring.

So we get: $m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n)$ (1)

Another way to derive is through Lagrangian:

$$L = \frac{m}{2} \sum_n \dot{u}_n^2 - \frac{k}{2} \sum_n (u_{n+1} - u_n)^2$$

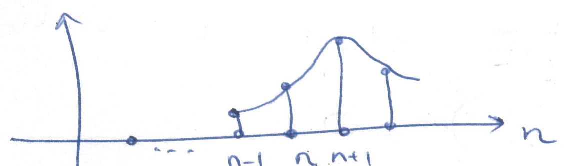
Euler-Lagrange equation: $\frac{d}{dt} \frac{\partial L}{\partial \dot{u}_n} - \frac{\partial L}{\partial u_n} = 0$, $n \in \mathbb{Z}$

$$m\ddot{u}_n + k(u_n - u_{n-1}) - k(u_{n+1} - u_n) = 0$$

$$m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n)$$

Assume "Slow" dependence on n:

$u_n(t) \approx u(x_n, t)$ is a smooth function



$u_{n+1} = u(x_{n+1}, t) = u(x_n + \ell, t) = u(x_n, t) + \ell \frac{\partial u}{\partial x}(x_n, t) + \frac{1}{2}\ell^2 \frac{\partial^2 u}{\partial x^2}(x_n, t) + O(\ell^3)$

$u_{n-1} = u(x_{n-1}, t) = u(x_n - \ell, t) = u(x_n, t) - \ell \frac{\partial u}{\partial x}(x_n, t) + \frac{1}{2}\ell^2 \frac{\partial^2 u}{\partial x^2}(x_n, t) + O(\ell^3)$

$u_n = u(x_n, t)$

Then

$$u_{n+1} + u_{n-1} - 2u_n = \ell^2 \frac{\partial^2 u}{\partial x^2}(x_n, t) + O(\ell^3)$$

$\Rightarrow$ So eq. (1) takes the form:

$$m \frac{\partial^2 u}{\partial t^2} = k\ell^2 \frac{\partial^2 u}{\partial x^2} + \text{l.o.t.}$$

Denoting $\frac{k\ell^2}{m} = c^2$ $\left(\text{so } c^2 = \frac{k\ell}{m/\ell} = \frac{\text{constant of spring of length}}{\text{density per unit length}}\right.$

we will call it sound speed.

Forgetting l.o.t, we get $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$.

Wave equation is universal!

Derivation 2 : [from symmetries and qualitative assymptions.]

space $x \in \mathbb{R}$, time $t \in \mathbb{R}_+$, $u(x,t) \in \mathbb{R}$
↑ state of the system

(H1) For any $u_0 \in \mathbb{R}$, the constant state $u(x) \equiv u_0$ is a stable equilibrium for any u_0

(H2) We consider small oscillations near $u(x,t) \equiv 0$

(H3) The system is homogeneous in space and time

(H4) Parity symmetry : $x \mapsto -x$ (In 3d it would corresp. left hand to right hand)

(H5) Time reversal symmetry : $t \to -t$

(H6) Long-wave approximations (oscillations are large-scale in space and time) ↑ are slowly changing in space/time

The general equation of motion for $u(x,t)$

$u(x,t) \xleftrightarrow{\text{Taylor series}} u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \dots$ at (x,t)

We are looking for a PDE that u satisfies

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \dots\right) = 0$$

(H3) $\Rightarrow$ $\mathcal{F}$ does not depend on (x,t)

(H2) $\Rightarrow$ we can linearize $\mathcal{F}$ at 0 $\left(u=0, \frac{\partial u}{\partial x}=0, \dots\right)$

$$C + C_{00}u + C_{10}\frac{\partial u}{\partial t} + C_{01}\frac{\partial u}{\partial x} + C_{20}\frac{\partial^2 u}{\partial t^2} + C_{11}\frac{\partial^2 u}{\partial t \partial x} + C_{02}\frac{\partial^2 u}{\partial x^2} + \dots = 0$$

(H1) $\xRightarrow{u_0=0}$ $C=0$

(H1) $u_0 \equiv \text{const} \neq 0$, $\frac{\partial u}{\partial x}=0$, $\frac{\partial u}{\partial t}=0, \dots$ $\Rightarrow$ $C_{00}u_0=0 \Rightarrow C_{00}=0$

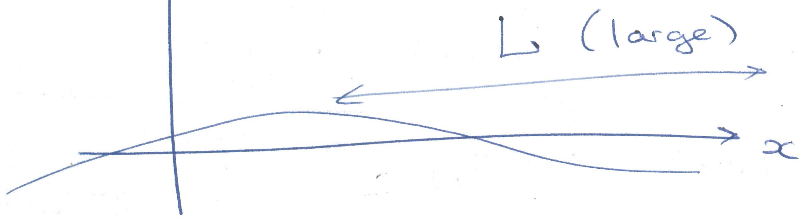
(H4) $\xRightarrow{x \mapsto -x}$ all odd derivatives w.r.t x change sign
$\Rightarrow$ no terms with odd-order derivatives w.r.t. x

(H5) $t \to -t$ $\Rightarrow$ no terms with odd-order derivatives w.r.t. t

$$C_{20}\frac{\partial^2 u}{\partial t^2} + C_{02}\frac{\partial^2 u}{\partial x^2} + C_{40}\frac{\partial^4 u}{\partial t^4} + C_{22}\frac{\partial^4 u}{\partial t^2 \partial x^2} + C_{04}\frac{\partial^4 u}{\partial x^4} + \dots = 0$$

(H6) Long-wave approximation: u changes on large scales L in x

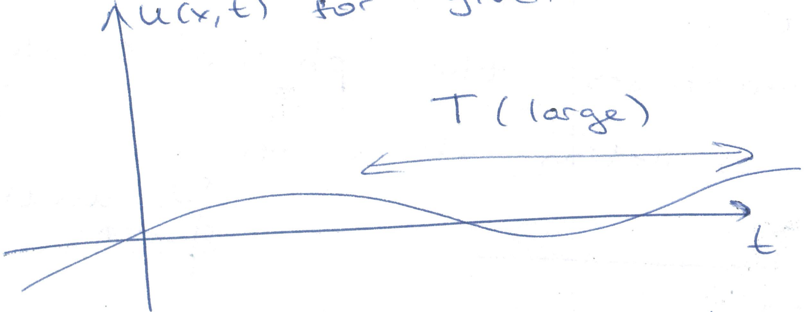
$u(x,t)$ for given t



and
u changes on large scales T in t

Change variables
$$\xi = \frac{x}{L}, \quad \tau = \frac{t}{T}$$

$u(x,t)$ for given x



$$\mathcal{U}(\xi,\tau) = u(x,t) = u(L\xi, T\tau)$$

$\Downarrow$

$\mathcal{U}(\xi,\tau)$ changes at characteristic scales $\xi \sim 1$, $\tau \sim 1$

$$\Rightarrow \quad \frac{\partial^{2m+2n}\mathcal{U}}{\partial \xi^{2m} \partial \tau^{2n}} \sim 1 \quad \text{for all } m,n$$

$$\frac{\partial^{2m+2n} u}{\partial x^{2m} \partial t^{2n}} \sim \frac{1}{T^{2m} L^{2n}} \underbrace{\frac{\partial^{2m+2n}\mathcal{U}}{\partial \xi^{2m} \partial \tau^{2n}}}_{1} \sim \frac{1}{T^{2m} L^{2n}}$$

$x = L\xi$ $\quad$ $t = T\tau$

In long-wave approximation we consider T and L large, so

the higher-order derivative terms are small
$\Rightarrow$ we can neglect terms with higher-order derivatives

$$\Rightarrow \quad C_{20}\frac{\partial^2 u}{\partial t^2} + C_{02}\frac{\partial^2 u}{\partial x^2} = 0 \qquad \left|\frac{C_{02}}{C_{20}}\right| = a c^2$$

$$\frac{\partial^2 u}{\partial t^2} \pm c^2 \frac{\partial^2 u}{\partial x^2} = 0.$$ What is the sign ~~of~~ before c?

If $\oplus$, then $\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^2 u}{\partial x^2} = 0.$

Then $u(x,t) = c e^{akt}\cos(kx)$ is a solution for any c and k!

$t=0$: $u(x,0) = c\cdot\cos(kx)$ if c is small,

But $u(x,t) \to \infty$ as $t\to\infty$

So you start with arbitrary small initial condition, you grow to infinity, so the $u=0$ is unstable equilibrium.

(H1) $\Rightarrow$ say $\ominus$ - sign $\Rightarrow \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$

We never used any physical origin!

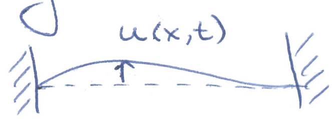
Observation: $x\in\mathbb{R}^3$ (H7) Isotropy in space

$\boxed{a=c}$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - a^2\Delta u = 0, \qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

a is what characterizes the physical / biolocical ... system

Examples: ① String: oscillation of a string



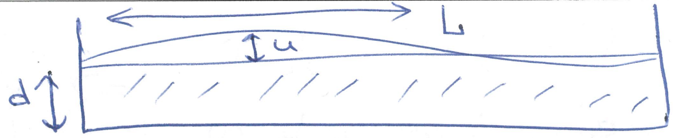
$a = \sqrt{T/\rho}$, T - tension, ρ - density. $u(x,t)$ - oscillation from an equilibrium

② Sound wave in gas or liquid
- u can be pressure or displacement of particles

Here a is a sound speed.

③ Electromagnetic wave
- u is a field (electric or magnetic)
- a is a light speed

④ Shallow water waves



- small amplitude
- "long" waves : $H \ll L$

depth is much smaller than the length of wave

$c = \sqrt{gH}$; $u(x,t)$ – displacement of the equilibrium

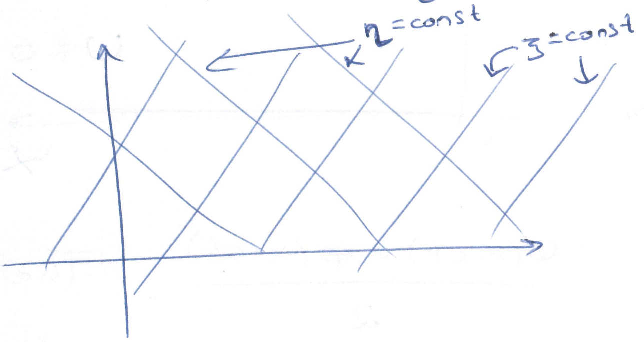
Well-posedness of Cauchy problem for 1D wave eq.

$$u_{tt} - c^2 u_{xx} = 0, \quad c \in \mathbb{R},\ x \in \mathbb{R},\ t \in \mathbb{R}_+$$

Let's find a general form of solution.

Make the change of variables: $\xi = x - ct$
$\eta = x + ct$

$u(x,t) = v(\xi, \eta)$



Using exercise 1, we get

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \xi}\right) = 0 \Rightarrow \frac{\partial v}{\partial \xi} = \underbrace{F(\xi)}_{\text{arbitrary}}$$

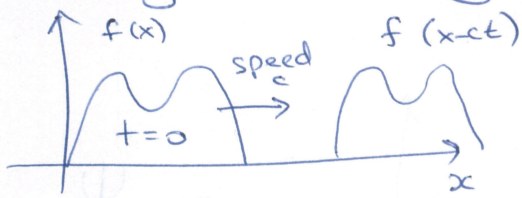
Integrate w.r.t. ξ $\Rightarrow$ $v(\xi,\eta) = \underbrace{\int F(\xi)d\xi}_{\text{arbitrary}} + \underbrace{g(\eta)}_{\text{arbitrary}}$

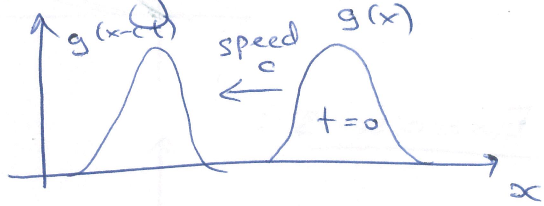
$\Rightarrow v(\xi,\eta) = f(\xi) + g(\eta)$ – general solution

Thus, $\boxed{u(x,t) = f(x-ct) + g(x+ct)}$

f represents wave moving to the right

g represents wave moving to the left





Well-posedness : 1) $\exists$ (existence)
2) ! (uniqueness)
3) continuous dependence on initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We will prove the $\exists$ and ! by providing an explicit formula for solution.

$u(x,0) = \varphi(x) \Rightarrow f(x) + g(x) = \varphi(x)$
$u_t(x,0) = \psi(x) \Rightarrow -cf'(x) + cg'(x) = \psi(x)$

$\Rightarrow \begin{array}{l} f' + g' = \varphi \\ -f' + g' = \frac{\psi}{c} \end{array}$

Thus, $f' = \frac{1}{2}\varphi' - \frac{1}{2c}\psi$ $\Rightarrow$ $\begin{cases} f = \frac{1}{2}\varphi - \frac{1}{2c}\int \psi(z)dz + C_1 \\ g = \frac{1}{2}\varphi + \frac{1}{2c}\int \psi(z)dz + C_2 \end{cases}$

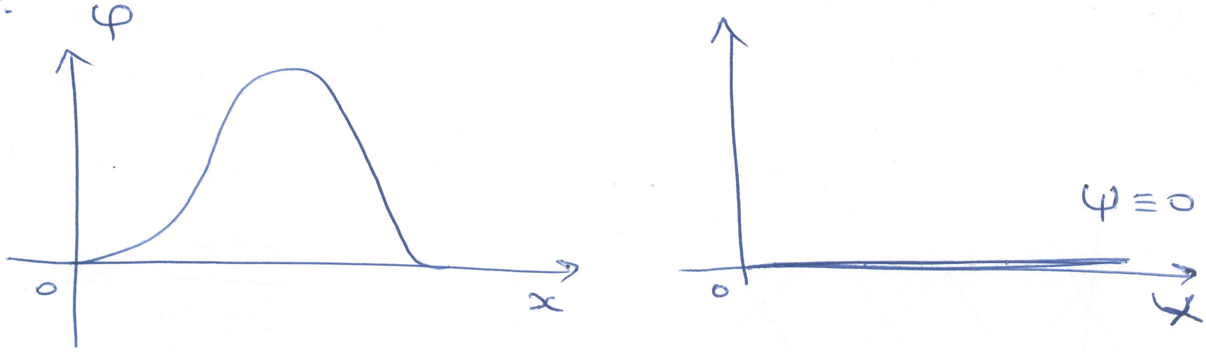
$g' = \frac{1}{2}\varphi' + \frac{1}{2c}\psi$

Note that $f + g = \varphi \Rightarrow C_1 + C_2 = 0$

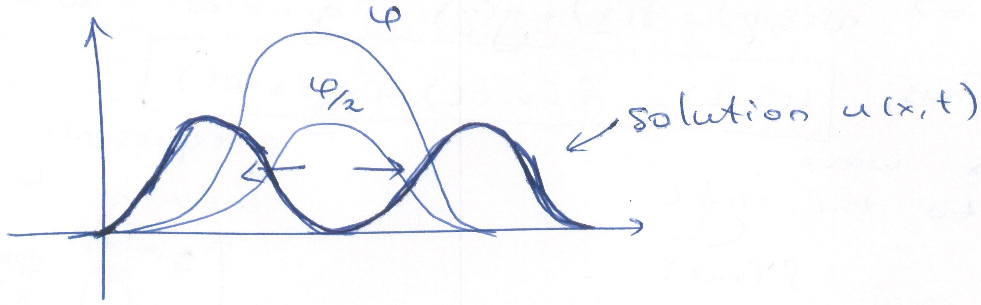
Then

$$\boxed{u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)dz}$$ D'Alambert formula

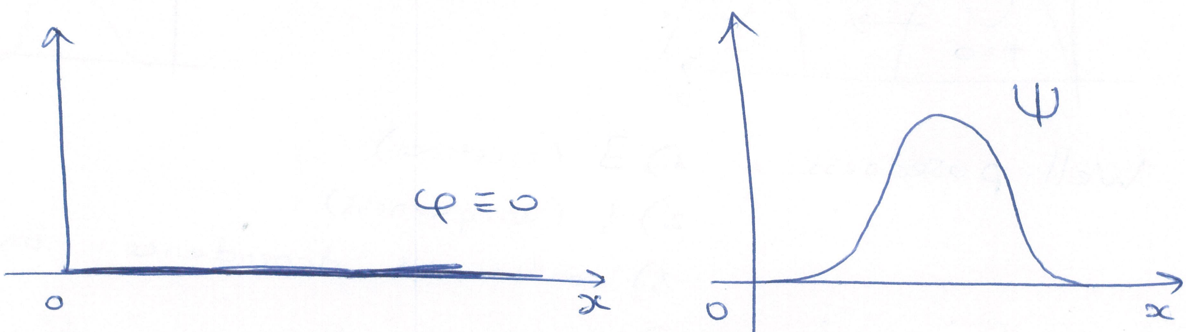
Example:



$\frac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$. Then $u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2}$. That means we will have exactly 2 waves with profiles $\frac{\varphi(x)}{2}$ going to left and right



Exercise 3:



Draw a solution u in this case.

Wave equation

Lecture 3

$$(*)\begin{cases} u_{tt} - c^2 u_{xx} = 0, & c \in \mathbb{R} \text{ - wave speed} \\ u(x,0) = \varphi(x) & x \in \mathbb{R},\ t>0 \\ u_t(x,0) = \psi(x) \end{cases}$$

Last time we proved that $\exists!$ solution to $(*)$. And derived D'Alambert formula for $u(x,t)$:

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)\,dz.$$

To finish the proof of well-posedness, we need to show the continuous dependence on initial data.

Remarks: for classical solution we see that we "need $u(x,t) \in C^2(\mathbb{R}\times[0,+\infty))$. So it makes sense to ask that $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$.

For simplicity, let us show the continuous dependence in the uniform norm $C(\mathbb{R})$, that is:

Lemma (cont. dependence in $C(\mathbb{R})$):

Let $\|\varphi - \varphi_1\|_{C(\mathbb{R})} < \varepsilon$ and $\|\psi - \psi_1\|_{C(\mathbb{R})} < \varepsilon$, $\varepsilon >$

and v is the solution of $(*)$ with $v(x,0) = \varphi($, $v_t(x,0) = \psi($

and v_1 is the solution of $(*)$ with $v_1(x,0) = \varphi_1($, $v_{1t}(x,0) = \psi_1($

Then, for any $T>0$ if $\varepsilon \to 0$ we have $v - v_1 \to 0$ uniformly in $x \in [0,T$

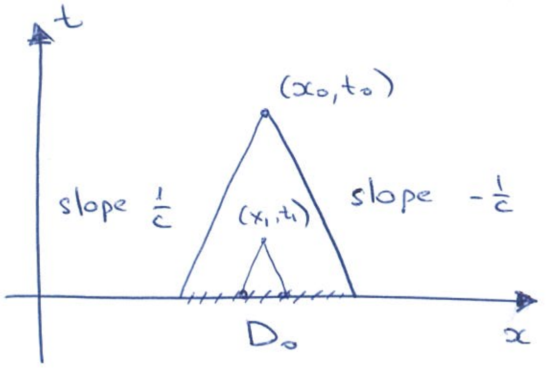
Proof:

$$\|v - v_1\|_{C(\mathbb{R})} \le \frac{\overbrace{|\varphi(x-ct) - \varphi_1(x-ct)|}^{<\varepsilon}}{2} + \frac{\overbrace{|\varphi(x+ct) - \varphi(x+ct)|}^{<\varepsilon}}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \overbrace{|\psi(z) - \psi_1(z)|}^{<\varepsilon}\,dz$$

Thus

$$\|v - v_1\|_{C(\mathbb{R})} \le \varepsilon(1+t) \le \varepsilon(1+T) \to 0 \quad \text{as } \varepsilon \to 0$$

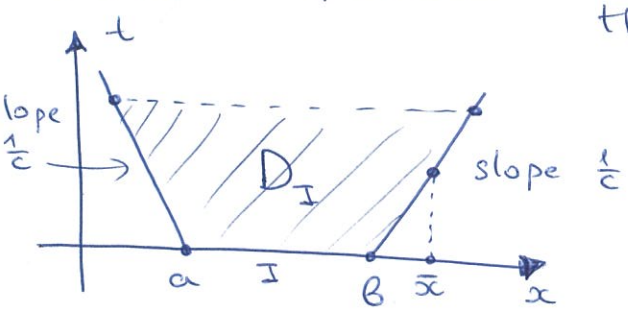
- <u>Domain of dependence</u> : $D_0 := \{x : x_0 - ct_0 < x < x_0 + ct_0\}$



So by D'Alambert formula we see that $u(x_0,t_0)$ depends only on values φ in points $(x_0 - ct_0)$ and $(x_0 + ct_0)$ and ψ on D_0. If we change φ and ψ outside D_0, the solution $u(x_0,t_0)$ will not change. That's why we call D_0 the domain of dependence.

Notice that for any point (x_1,t_1) inside triangle the domain of dependence is also inside ~~a triange~~ D_0.

- <u>Reversed question</u> : what points are influenced by the data in an interval I on $t=0$



$I = [a,b]$

D_I - domain of influence of I

$D_I = \{(x,t) : t \in [0,T]$ and $a - ct \leq x \leq b + ct\}$

We say that disturbances propagate at speed c

We mean the following:

Let φ and ψ be supported on I ($\varphi = 0$, $\psi = 0$ out of I)

Imagine the observer is at point $\bar{x} \notin I$, say $\bar{x} > b$

For all times $t < \frac{\bar{x} - b}{c}$ the solution u will be 0 (the observer doesn't feel the disturbance). However, once $t \geq \frac{\bar{x} - b}{c}$ the solution will depend on φ, ψ forever!

<u>Remark</u> : interesting observation that we do not touch in these lectures:

think of the sound! → in $\mathbb{R}^3$ (in fact in $\mathbb{R}^{2d+1}$, $d \in \mathbb{N}$) if we hear some signal we start to hear it and finish to hear it at some point, that is the solution $\exists\, t_1 < t_2$ s.t.

$u(x,t) = 0$, if $t < t_1$ and $u(x,t) = 0$ if $t > t_2$.

As we see in $\mathbb{R}^1$ this is not the case!

Also it is not the case for $\mathbb{R}^{2d}$, $d \in \mathbb{N}$.

Inhomogeneous wave equation:

$$(**)\quad \begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \qquad \begin{array}{l} h \in C(\mathbb{R} \times [0,+\infty)) \\ \varphi \in C^2 \\ \psi \in C^1 \end{array}$$

→ Use linearity: consider

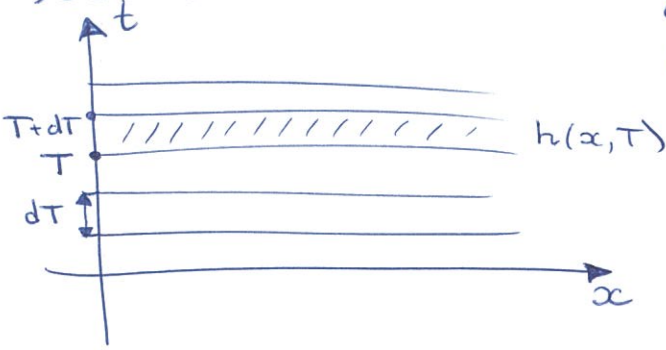
$$(1)\ \begin{cases} (u_1)_{tt} - c^2 (u_1)_{xx} = h(x,t) \\ u_1(x,0) = 0 \\ (u_1)_t(x,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (u_2)_{tt} - c^2 (u_2)_{xx} = 0 \\ u_2(x,0) = \varphi(x) \\ (u_2)_t(x,0) = \psi(x) \end{cases}\ (2)$$

then $u(x,t) = u_1(x,t) + u_2(x,t)$

We know how to solve (2). How to solve (1)?
Let me give you "a general construction" that allows to solve (1) if you know how to solve (2).
It is called <u>Duhamel principle</u>.

The idea is to move $h(x,t)$ from RHS (right hand side of (1)) to the initial data in (2).

<u>Intuition</u> (physical): the term $h(x,t)$ acts as an external force at every point x in space and time t.



Let's divide the (x,t)-place into strips of infinitesimal lengths dT, and assume the forcing there is constant

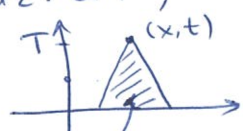
$$h(x,T) \Rightarrow \begin{array}{l} u_{tt} \sim h(x,T) \\ u_t \sim h(x,T)\,dT \end{array}$$

So we can consider an auxiliary problem:

$$\begin{cases} \mathcal{U}_{tt} - c^2 \mathcal{U}_{xx} = 0 \\ \mathcal{U}(x,T) = 0 \\ \mathcal{U}_t(x,T) = h(x,T)\,dT \end{cases} \underset{\text{D'Alamb.}}{\Rightarrow} \mathcal{U}(x,t) = \frac{1}{2c} \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz \cdot dT$$

If we now consider the time from $t=0$ to $t=T+dT$, that we need to sum up all $\frac{1}{2c} \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz \cdot dT$, that is to take integral:

$$\frac{1}{2c} \int_0^t \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz\,dT$$



this is the

Let's prove this mathematically rigorous.

Duhamel principle: take $v = v(x,t;s)$ such that

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & t > s \\ v(x,t;s) = 0, & t = s \\ v_t(x,t;s) = f(x,s), & t = s \end{cases}$$

Then $u(x,t) = \int_0^t v(x,t;s)\,ds$ is a solution to (1)

Proof:

$$u_t = \cancel{v(x,t;t)}^{\,0} + \int_0^t v_t(x,t;s)\,ds$$

$$u_{tt} = v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,ds =$$

$$= f(x,t) + \int_0^t v_{tt}(x,t;s)\,ds$$

$$u_{xx} = \int_0^t v_{xx}(x,t;s)\,ds$$

Then $u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t \underbrace{\left(v_{tt}(x,t;s) - c^2 v_{xx}(x,t;s)\right)}_{=0} ds$ ■

There is an exercise 3 to solve inhomogeneous wave equation in a different manner (using Green's theorem).

Thus, the solution to (**) looks like:

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z,t)\,dz + \frac{1}{2c}\int_0^t \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz\,dT.$$

Remark: Duhamel principle is a powerful (universal) method of solving inhomogeneous problems. It works for ODEs, heat equation etc....

Mixed initial-boundary value problem

Consider a string of a guitar

$$u_{tt} - c^2 u_{xx} = h(x,t), \quad x \in [a,b]$$

$$\left.\begin{aligned} u(x,0) &= \varphi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}\right\} \text{"initial" conditions} \qquad (***)$$

$$\left.\begin{aligned} u(a,t) &= a(t) \\ u(b,t) &= b(t) \end{aligned}\right\} \text{"boundary" conditions}$$

One can solve this problem explicitly using Fourier sums (we will do it later). But let us show that even if we do not know the exact form of solution, we can prove $\exists$ and $!$

Thm (uniqueness for wave equation)

There exists at most one function $u \in C^2([a,b] \times [0,T$ solving $(***)$.

Proof:

Γ We will prove using "energy method".

Suppose w, v are two solutions of $(***)$

Then $u = w - v$ is a solution to homogeneous problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(a,t) = u(b,t) = 0. \end{cases}$$

Let us show that $u \equiv 0$.

Define the "energy":

$$I(t) = \frac{1}{2} \int_a^b \big(\underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{u_t^2} + \underset{\substack{\uparrow \\ \text{potential} \\ \text{energy}}}{c^2 u_x^2} \big)\, dx$$

How does $I(t)$ change with time?

$$\frac{dI(t)}{dt} = \frac{1}{2} \int_a^b \big(2 u_t \cdot \underbrace{u_{tt}}_{= c^2 u_{xx}} + c^2 \cdot 2 \cdot u_x \cdot u_{xt} \big)\, dx =$$

$$= c^2 \int_a^b \left(u_t \cdot u_{xx} + u_x \cdot u_{xt} \right) dx = c^2 \int_a^b \frac{d}{dx} (u_t \cdot u_x)\, dx$$

$$= c^2 u_t \cdot u_x \Big|_a^b = 0 \implies I(t) \equiv \text{const}$$

$I(0) = 0 \implies I(t) \equiv 0$. Thus $u_t \equiv 0$, $u_x \equiv 0 \implies u \equiv \text{con}$

As $u(x,0) = 0 \implies u \equiv 0$

Thm (existence of solution to a wave equation)

There exists a solution to problem $(***)$ $u \in C^3([a,b] \times [0,T])$

Proof:

For simplicity, let $c=1$ (the same thing for $c \neq 1$, let it be an exercise)

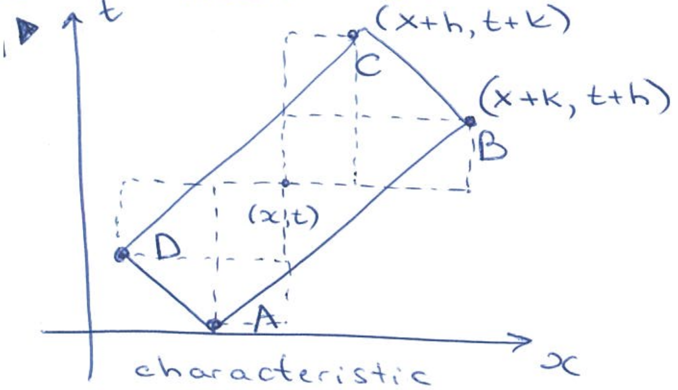
Before we prove, let me formulate and prove

Useful lemma: $u(x,t) \in C^3$. The following statements are equivalent:

(1) u satisfies PDE $u_{tt} - u_{xx} = 0$

(2) u satisfies the difference equation
$$u(x-k, t-h) + u(x+k, t+h) = u(x-h, t-k) + u(x+h, t+k)$$
$\forall (x,t) \in \mathbb{R} \times \mathbb{R}$ and $k, h > 0$. See remark below[1].

Proof of lemma:



characteristic rectangle

$\boxed{(1) \Rightarrow (2)}$ $u(x,t) = f(x-t) + g(x+t)$

Notice that f and g separately satisfy the difference eq: $f(x-k-t+h) + f(x+k-t-h)$
$= f(x-h-t+k) + f(x+h-t-k)$
$\Rightarrow u$ satisfies the diff. eq.

$\boxed{(2) \Rightarrow (1)}$ Let $h=0$.

$u(x,t)$ satisfies the difference equation in (2). Subtrack $2u(x,t)$ and devide for k^2:

$$\frac{u(x-k,t) - 2u(x,t) + u(x+k,t)}{k^2} = \frac{u(x,t-k) - 2u(x,t) + u(x,t+k)}{k^2}$$

By Taylor expansion, we get

$$u(x-k,t) = u(x,t) - k u_x(x,t) + \tfrac{1}{2} k^2 u_{xx}(x,t) + O(k^3)$$ here we use that u is from C^3

$$u(x+k,t) = u(x,t) + k u_x(x,t) + \tfrac{1}{2} k^2 u_{xx}(x,t) + O(k^3)$$

So we have $u_{xx} + O(k) = u_{tt} + O(k)$, $k \to 0$

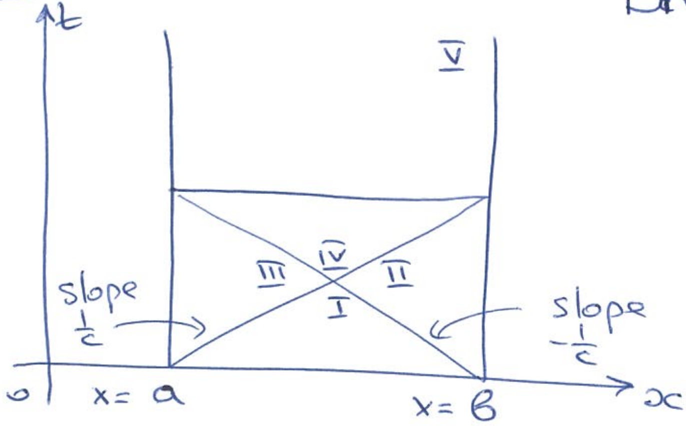
As a limit we get the wave equation. ∎

---

[1] Intuitively formula (2) is very clear. Indeed the left hand side is a discrete analog of u_{xx} (if $h=0$): $u(x-k,t) + u(x+k,t) =$
$= (u(x-k,t) - u(x,t)) - (u(x,t) - u(x+k,t)) \sim -u_x(x-k) + u_x(x) \sim u_{xx}(x)$

Proof of existence: simple geometric idea.

$c=1$

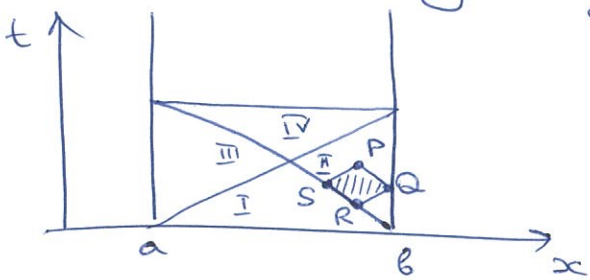


Divide the domain $\Omega = [a,b] \times \mathbb{R}_+$ into 5 pieces as shown on the picture: draw a line with slope $\frac{1}{c}$ from point a, and a line with slope $-\frac{1}{c}$ from point b, and consider a rectangle such that these two lines are its diagonals.

Then the following observations are valid.

I. The solution in region I is completely determined by D'Alembert formula.

II. To construct solution at any point P in region II we use the following characteristic rectangle (see picture) and use useful lemma.



$$u(P) + u(R) = u(S) + u(Q)$$

boundary cond.

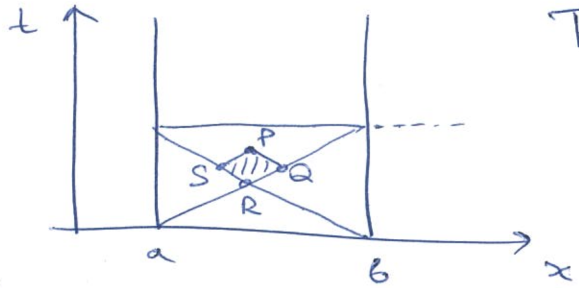
$$\Rightarrow u(P) = u(S) + u(Q) - u(R)$$

we already know!

Thus we know u in region II.

III. Analogously, we construct u in region III.

IV. To construct u in region IV we use the following characteristic rectangle and use useful lemma.



Thus, we have constructed the solution for $x \in [a,b]$, $t \in [0, \frac{b-a}{c}]$.

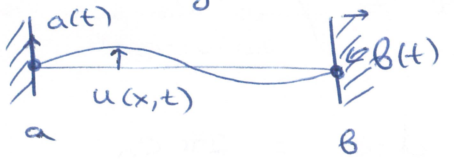
Repeat this procedure to construct u for all $t>0$ ∎

Exact solution to :

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_{x=0} = u|_{x=\pi} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Lecture 4 Last time: mixed initial-boundary value problem

guitar string oscillation



$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(a,t) = a(t) \\ u(b,t) = b(t) \end{cases}$$

(ic) for the two initial conditions, (bc) for the two boundary conditions.

- We proved $\exists!$ of solution
- $c = \sqrt{T/\rho}$, ρ-density, T-tension

Today let us find explicitly the solution to:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(a,t) = u(b,t) = 0. \end{cases}$$

We will do it using Fourier series.

Small reminder on Fourier series

Def: Fourier series of function f is a representation:

$$(1) \quad \boxed{f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}}, \quad c_n \in \mathbb{C}, \quad x \in \mathbb{R}, \quad f: \mathbb{R}\to\mathbb{R}$$

(in fact, f is periodic with period 2π)

Here the series converges absolutely. It is enough to assume that $\sum_{n\in\mathbb{Z}} |c_n| < +\infty$

Rmk: if f is real $\Rightarrow$ $f(x) = \overline{f}(x) = \sum_{n\in\mathbb{Z}} \overline{c}_n e^{-inx} = \sum_{n\in\mathbb{Z}} \overline{c}_{-n} e^{inx}$

$$\Rightarrow \quad \boxed{\overline{c}_{-n} = c_n}$$

$\Rightarrow$ let's define $c_0 = \frac{a_0}{2}$, $c_n = (a_n - ib_n)\frac{1}{2}$

$c_{-n} = (a_n + ib_n)\frac{1}{2}$

Thus, $f(x) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(c_n e^{inx} + c_{-n} e^{-inx}\right) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(\frac{a_n}{2}\cos(nx) + \frac{b_n}{2}\sin(nx)\right) + \sum_{n\in\mathbb{N}} \left(\frac{a_n}{2}\cos(nx) + \frac{b_n}{2}\sin(nx)\right)$

$$\Rightarrow \quad \boxed{f(x) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left[a_n\cos(nx) + b_n\sin(nx)\right]} \qquad (2)$$

Thm 1: For a function given by Fourier series (1), we can define a coefficient:

Fourier coef. $\quad \boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx} \qquad (3)$

Proof:

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}} c_m e^{imx} e^{-inx}\,dx = \Big/ \text{can change } \int \text{ and } \sum \text{ as the sum is absolutely convergent} \Big/$$

$$= \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = 2\pi c_n$$

because $\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 2\pi, & m=n \\ 0, & m\neq n \end{cases}$ ■

Observation:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$$

Remark: from point of view of functional analysis:

consider a Hilbert space

$$L^2[0,2\pi] = \Big\{ f: [0,2\pi]\to\mathbb{C} \text{ - measurable}: \int_0^{2\pi} |f(x)|^2 dx < +\infty \Big\} \Big/ \sim$$

where $f\sim g$ means $f=g$ w.r.t. Lebegue measure

that is $\mu\{x: f(x)\neq g(x)\} = 0$.

μ – Lebegue measure on $\mathbb{R}$.

Then • $\left(\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2} = \|f\|_{L^2[0,2\pi]}$ - norm

• $\int_0^{2\pi} f(x)\overline{g(x)}\,dx = \langle f,g\rangle_{L^2[0,2\pi]}$ - scalar product

Actually, $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal basis

($\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormall basis)

And $\forall f\in L^2[0,2\pi]$ can be represented by formula (1)

$$\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \delta_{nm} \leftarrow \text{Kronecker symbol}$$

$$\delta_{nm} = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

in the sense of L_2: $\int_0^{2\pi} \Big| f(x) - \sum_{|n|\leq N} c_n e^{inx} \Big|\,dx \xrightarrow[N\to\infty]{} 0$

In finite dimensions we have $u \in \mathbb{R}^n$ and $\{e_1, .., e_d\}$ basis, then $\exists!\ u_k$: $u = \sum_{k=1}^{d} u_k e_k$

To find u_k we just take scalar product with e_n

$$\langle u, e_n \rangle = \sum_{k=1}^{d} u_k \cdot \langle e_k, e_n \rangle = u_n \langle e_n, e_n \rangle$$

$$\Rightarrow \quad u_n = \frac{\langle u, e_n \rangle}{\langle e_n, e_n \rangle}.$$

For infinite dimensional space it is similar.

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \quad \Big| \cdot \ \langle \cdot, e^{imx} \rangle$$

$$\langle f(x), e^{imx} \rangle = c_m \underbrace{\langle e^{imx}, e^{imx} \rangle}_{= 2\pi}.$$

$$\Rightarrow \quad c_m = \frac{1}{2\pi} \langle f(x), e^{imx} \rangle.$$

- The same story for $\{1, \cos(nx), \sin(nx)\}$ - basis in $L^2[0, 2\pi]$ for real-valued f.

<u>Thm 2:</u> Let $f(x) \in C^\infty(S^1)$ - smooth periodic function on a circle $S^1 = [0, 2\pi] / \{0 = 2\pi\}$

Then for any $a \geq 0$ there exists a constant C (which depend on f and a, but independent of n) such that

$$|c_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \right| \leq C \cdot |n|^{-a} \quad \text{for } |n| \neq 0$$

(c_n goes to 0 very fast - faster than any polynomial

Proof :

▶ $a = 0$: $$|c_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) e^{-inx}| \, dx = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| \, dx}_{=: C}$$ proved!

$a = 1$: integrate by parts :

$$|c_n| = \left| \cancel{\frac{1}{2\pi} f(x) \frac{e^{-inx}}{-in} \Big|_{-\pi}^{\pi}}^{\,0} + \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} \, dx \right| \leq$$

$$\leq \frac{1}{n} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)| \, dx =: \frac{C}{n}.$$ proved!

And so on....

Corollary: For any $f(x) \in C^\infty(S^1)$ the corresponding Fourier series

$$\sum_{n\in\mathbb{Z}} c_n e^{+inx}, \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,$$

converges for all $x \in \mathbb{R}$.

Proof:

$\ulcorner$ Absolute convergence is clear:

$$\left|\sum c_n e^{inx}\right| \le \sum |c_n| \underbrace{\left|e^{inx}\right|}_{=1} \le \sum \frac{c}{n^2} < +\infty \quad \blacksquare$$

$\llcorner$

Summing up:

Thm 1: $f(x) = \sum c_n e^{inx} \Rightarrow c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,d$

Thm 2 + Corrollary: $f \in C^\infty(S^1) \Rightarrow$ we can write a series $\sum c_n e^{-inx}$ and it converges.

Does this series always converge to $f(x)$?

Not always! (in general)

Thm 3 (without proof): $f \in C^2(S^1)$, then $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$, $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,d$

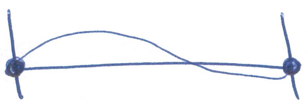
e.g. Arnold's book

Solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u\big|_{x=0} = u\big|_{x=\pi} = 0 \\ u\big|_{t=0} = \varphi(x) \\ u_t\big|_{t=0} = \psi(x) \end{cases}$$

$\varphi(0) = \varphi(\pi) = 0$

$\psi(0) = \psi(\pi) = 0$



0 $\quad\quad$ π

Spectral method: we have constant coefficients, let's try solution of the form:

(we will need to solve auxiliary eigenvalue problem)

$$u(x,t) = \varphi(x)\cdot e^{\lambda t} \quad \text{for some } \lambda \in \mathbb{C}$$

$$\lambda^2 \cdot \varphi(x) \cdot \cancel{e^{\lambda t}} - c^2 \cdot \varphi''(x) \cdot \cancel{e^{\lambda t}} = 0$$

So we have $\begin{cases} \varphi'' + \mu\varphi = 0 \\ \varphi(0) = \varphi(\pi) = 0 \end{cases}$ for $\mu = -\frac{\lambda^2}{c^2} \in \mathbb{C}$ $\to$ eigenvalue problem

$$\begin{cases} \varphi'' + \mu\varphi = 0 \\ \varphi(0) = \varphi(\pi) = 0 \end{cases}$$ Let's find all μ for which the solution exists.

Case $\mu < 0$: $\varphi'' + \mu\varphi = 0 \Rightarrow \varphi(x) = A e^{\sqrt{|\mu|}x} + B e^{-\sqrt{|\mu|}x}$

$$\begin{matrix}\varphi(0)=0 \\ \varphi(\pi)=0\end{matrix} \Rightarrow \begin{cases} A + B = 0 \\ A \cdot e^{\sqrt{\mu}\pi} + B e^{-\sqrt{\mu}\pi} = 0 \end{cases} \Rightarrow A = B = 0$$

only trivial solution $\varphi \equiv 0$.

Case $\mu = 0$: $\varphi'' = 0 \Rightarrow \varphi(x) = Ax + B$

$\varphi(0) = \varphi(\pi) = 0 \Rightarrow A = B = 0 \Rightarrow \varphi \equiv 0$.

Case $\mu > 0$: $\varphi'' + \mu\varphi = 0 \Rightarrow \varphi(x) = A \cdot e^{i\sqrt{\mu}x} + B \cdot e^{-i\sqrt{\mu}x}$

better $\varphi(x) = A\sin(\sqrt{\mu}x) + B\cos(\sqrt{\mu}x)$

$\varphi(0) = 0 \Rightarrow B = 0 \Rightarrow \varphi(x) = A\sin(\sqrt{\mu}x)$

$\varphi(\pi) = 0 \Rightarrow A\sin(\sqrt{\mu}\pi) = 0 \Rightarrow \sqrt{\mu}\pi = \pi k,\ k \in \mathbb{Z}$

$\mu = k^2,\ k \in \mathbb{Z}$

Obs: μ can be only real and positive

$$\varphi'' + \mu\varphi^* = 0 \ \Big|\ \cdot \langle \cdot, \varphi \rangle_{L^2}$$

$$\int_0^{2\pi} \varphi'' \cdot \varphi + \mu\varphi^2 = 0 \qquad \mu \cdot \int_0^{2\pi} \varphi^2 = \int_0^{2\pi} (\varphi')^2$$

$$\mu = \frac{\int_0^{2\pi} (\varphi')^2}{\int_0^{2\pi} \varphi^2} > 0.$$

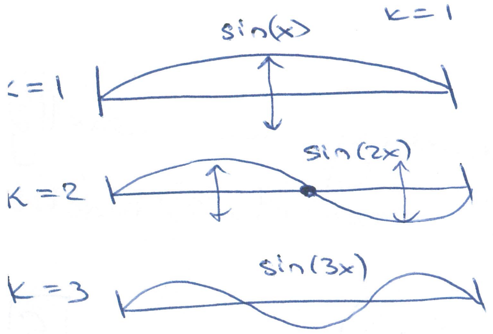
Then $\lambda^2 = -c^2k^2$

$\lambda = ick$ and we have infinitely many

solutions: $u_k(x,t) = \sin(kx) \cdot \underbrace{e^{ikt}}_{\text{complex}}$

we are interested only in real solutions, so we can consider any sums like this:

$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left[A_k \cos(ckt) + B_k \sin(ckt)\right]$$

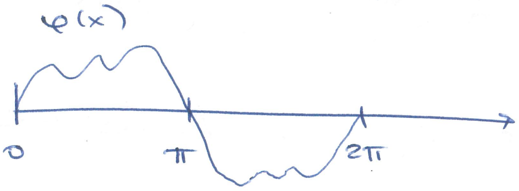


main mode (fundamental tone)

overtone

Let us show that this solution is general.

First, notice that $\varphi(x)$ can be represented only as a sum of $\sin(kx)$ in its Fourier series.



→ continue $\varphi(x)$ to interval $[\pi, 2\pi]$ oddly $\Rightarrow$ $$\varphi(x) = \sum_{k=1}^{\infty} A_k \sin(kx)$$

→ similar with $\psi(x)$:

$$\psi(x) = \sum_{k=1}^{\infty} B_k \sin(kx)$$

Second, we have $$u(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left(A_k \cos(kt) + B_k \sin(kt)\right)$$

$$u(x,0) = \varphi(x) = \sum A_k \sin(kx) \Rightarrow A_k = a_k$$

$$u_t(x,0) = \psi(x) = \sum ck B_k \sin(kx) \Rightarrow ckB_k = b_k \Rightarrow B_k = \frac{b_k}{kc}.$$

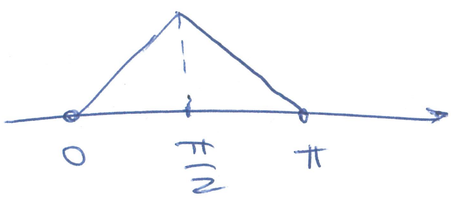
$\Rightarrow$

$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left(a_k \cos(kt) + \frac{b_k}{kc}\sin(kt)\right).$$

where $a_k = \frac{2}{\pi}\int_0^{\pi} \varphi(x)\sin(kx)\,dx$, $b_k = \frac{2}{\pi}\int_0^{\pi} \psi(x)\sin(kx)\,d$

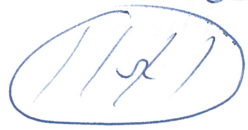
Exercise: Find a Fourier series solution to

$$\varphi(x) = \begin{cases} x, & x \in [0, \frac{\pi}{2}] \\ \pi - x, & x \in [\frac{\pi}{2}, \pi] \end{cases}$$



$$\psi(x) \equiv 0.$$

Various space dimensions: $\Omega \subset \mathbb{R}^d$



$$u_{tt} - c^2 \Delta u = 0 \qquad u|_{\partial\Omega} = 0$$

One can look for solutions of the form:

$u(r,t) = \varphi(r) \cdot e^{i\omega t}$ and have the following eigenvalue problem:

$$\begin{cases} \Delta\varphi + \frac{\omega^2}{c^2}\varphi = 0 \\ \varphi|_{\partial\Omega} = 0 \end{cases}$$

For compact Ω with smooth boundary, we usually have a family $\{\frac{\omega_k^2}{c^2}, \varphi_k\}_{k\in\mathbb{N}}$ of eigenvalues/functio

$$u(x,t) = \sum_{k} \varphi_k(x)\left(A_k \sin(c\omega_k t) + B_k \cos(c\omega_k t)\right).$$