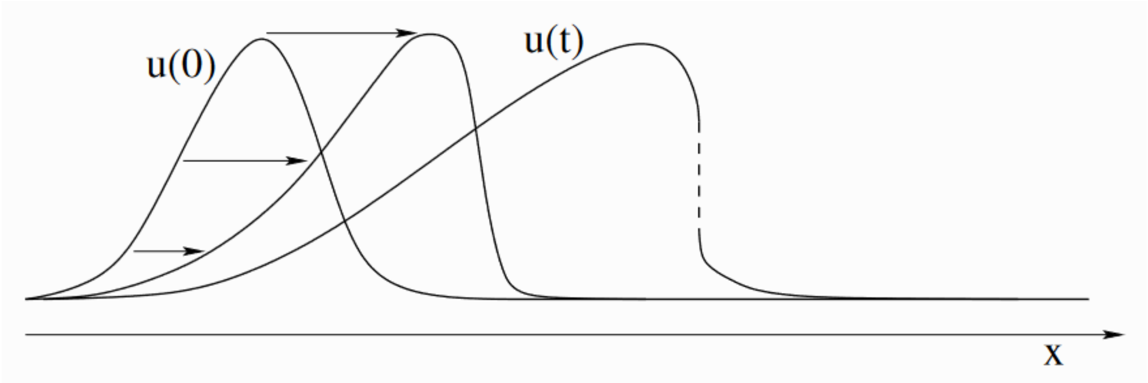
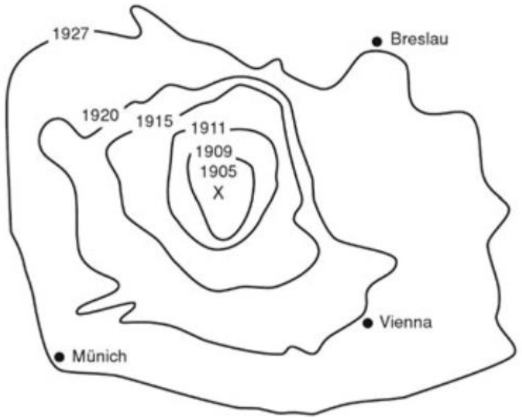
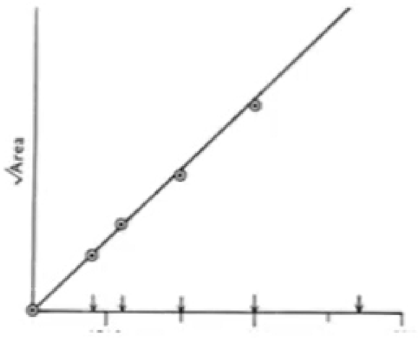
Shock waves in conservation laws and reaction-diffusion equations







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Any questions are very welcome!
Group in Telegram: https://t.me/+gdqus0TfJQ44NWI6

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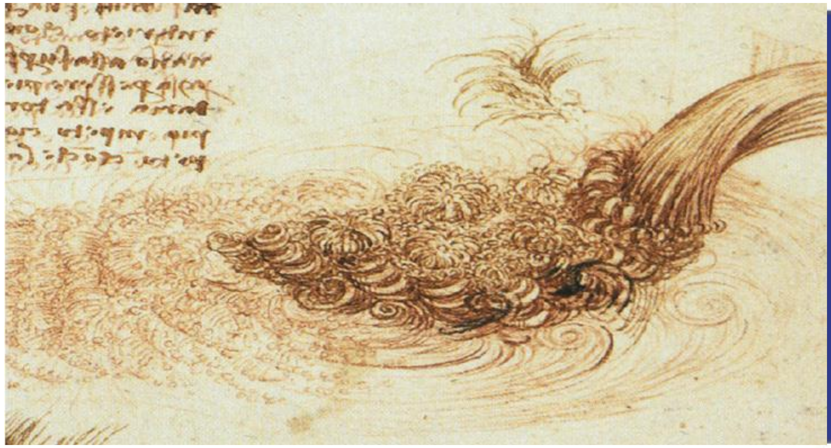
Matemática PUC-Rio

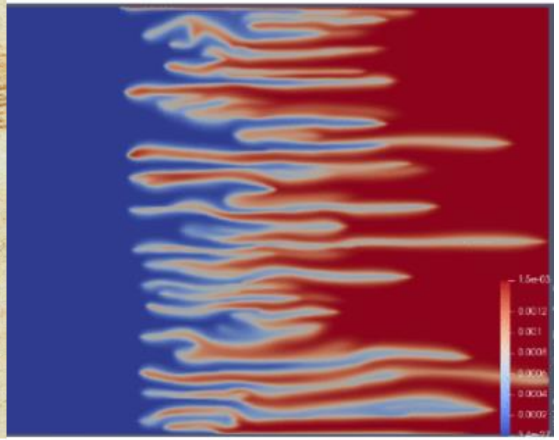
Motivation

Many phenomena in “nature” can be described using mathematical tools:

1. Physics (classical):
   - Mechanics, thermodynamics, fluid dynamics, electrodynamics
2. Biology and social sciences:
   - Population dynamics
     – how animals / bacteria / viruses / tumours spread?
   - Pattern formation
     – why do lizards have such a skin?
     – why do birds fly forming a triangle?



Leonardo Da Vinci describes turbulent motion of water (around 1500)

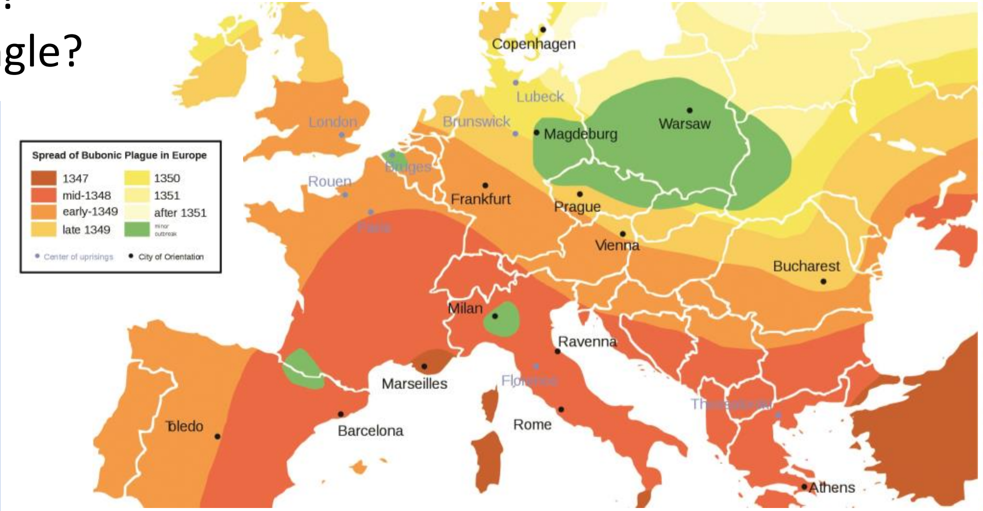


Oil recovery: displacement of oil by water

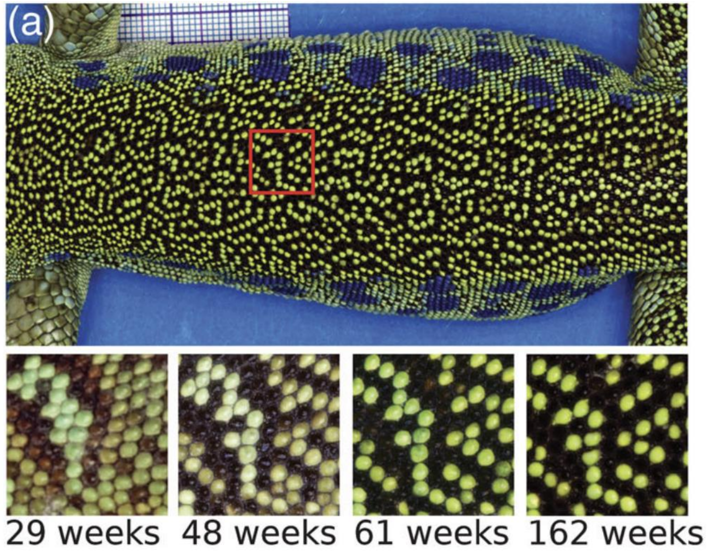
Basic idea:
- Create a mathematical “model”
- Study the properties of its “solutions”

One of the conventional tools is:
PDE (partial differential equation)

Not the only one!
Probability, algebraic geometry etc…



Spread of Bubonic Plague in Europe (around 1350)



What is a PDE? First example: $\Delta T = 0$

Let $T(x,t)$ be a temperature in the classroom. Here $x \in \Omega \subset \mathbb{R}^3$, $t \in \mathbb{R}_+$.

- In equilibrium:

$$\int_{\partial V} \vec{F} \cdot \nu \, dS = 0$$

$\vec{F}$ - heat flux.

- Use Green-Gauss theorem:

$$\int_{\partial V} \vec{F} \cdot \nu \, dS = \int_V div(\vec{F}) \, dx$$

- As this is true for all domains V, we get $div(\vec{F}) = 0.$
- Assume heat flux is proportional to gradient of temperature:

$$\vec{F} = -a \, \nabla T$$

(the more is the difference of the temperature between points, the faster is the heat flow)

Finally, we get:

$$div(\nabla T) = \Delta T = 0 \qquad \text{(Laplace equation)}$$

Pierre-Simon Laplace
(1749 – 1827)

What do you need to set up a PDE problem?

(1) Fix a domain $\boldsymbol{x} = (\boldsymbol{x_1}, \dots, \boldsymbol{x_n}) \in \boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{n}}$ and consider an equation for an unknown function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ for $\boldsymbol{x} \in \boldsymbol{\Omega}$:

$$P\left(x, u(x), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}}\right) = 0$$

Augustin-Louis Cauchy (1789-1857)

The order of the highest derivative $k \in \mathbb{N}$ is called the order of the PDE.
If $n = 1$, then it is called ODE (ordinary differential equation), otherwise PDE.

(2) Fix additional boundary or initial conditions on (possibly a part) of $\boldsymbol{\partial\Omega}$.

Caution: for ODEs one "typically" considers the so-called Cauchy problem:

$$u'' = f\big(t, u(t)\big)$$
$$u(0) = u$$
$$u'(0) = v$$

For PDEs the situation is more tricky and more elaborate conditions often should be considered.

(3) Fix to which functional space the function $\boldsymbol{u}$ belongs.

It may be $C(\Omega), C^k(\Omega)$ or some weaker spaces like $L_2(\Omega)$, Sobolev space or BV functions (functions of bounded variation)
Another thing could be that one assumes different smoothness requirements for different variables (e.g. if one of the variables corresponds to time)

Typical questions of mathematical interest:

(1) Well-posedness (in Hadamard sense, around 1902)

a. The solution exists (∃)
b. The solution is unique (!)
c. There is a continuous dependence of the solution on the "initial"/"boundary" data

- Ill-posed problems – we will see in a course

Jacques Hadamard
(1865 – 1963)

(2) Qualitative properties of the solution:

– How does the solution look like?
– Does there exist a solution of special type? E.g. having some symmetries.

- If the problem is evolutionary (there is a time variable), then a natural question is:
  – What is a long-time behaviour of the solution as $t \to \infty$?

Remark:
from my experience working with engineers the questions of existence and uniqueness are not so important for them, but the continuous dependence, indeed, is important. The reason is that there is also some noise (in the measurements, modelling etc), so it can cause big problems for them if the small change in initial data lead to big changes in solution.

A ZOO of PDEs (see Evans book on PDEs for more examples)

Linear PDEs:

(Laplace equation) $\Delta u = 0$

(Heat equation) $u_t = \Delta u$

(Linear transport equation) $u_t + \sum_{n=1}^{k} c_i u_{x_i} = 0$

(Schrodinger equation) $i u_t + \Delta u = 0$

(Wave equation) $u_{tt} - \Delta u = 0$

And many more….

Non-linear PDEs (and systems):

(Inviscid Burgers equation) $u_t + \left(\frac{u^2}{2}\right)_x = 0$

(Scalar conservation law) $u_t + div\,(F(u)) = 0$

(Scalar reaction-diffusion equation) $u_t = \Delta u + f(u)$

(Euler equation)

$$u_t + (u \cdot \nabla)\, u = \nabla p$$
$$\nabla \cdot u = 0$$

(Navier-Stokes equation)

$$u_t + (u \cdot \nabla)\, u - \nu \Delta u = \nabla p$$
$$\nabla \cdot u = 0$$

And many more….

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And many more….

Typical principles from Evans book on PDEs

1. Nonlinear equations are more difficult than linear equations; and, indeed, the more the nonlinearity affects the higher derivatives, the more difficult the PDE is.

2. Higher-order PDE are more difficult than lower-order PDE

3. Systems are harder than single equations

4. PDEs entailing many independent variables are harder that PDEs entailing few independent variables

5. For most PDEs it is not possible to write out explicit formulas for solutions

None of these assertions is without important exceptions.

Four main PDEs in our course:

1. **Transport equation:**

$$u_t + c\, u_x = 0$$

2. **Wave equation:**

$$u_{tt} - c^2\ u_{xx} = 0$$

3. **Scalar conservation law:**

$$u_{\mathrm{t}} + \big(F(u)\big)_x = 0$$

4. **Reaction-diffusion equation:**

$$u_t = u_{xx} + f(u)$$

They are all different (linear/non-linear), require different mathematical tools to be analysed,

BUT

Solution to these equations exhibit a “propagation” phenomena:

there are “waves” that are moving

P.S. I write the simplified version, that is for $x \in \mathbb{R}$, $u \in \mathbb{R}$, there exist various generalisations.

Transport equation

$$u_t + c\,u_x = 0$$
$$u(x,0) = u_0(x)$$

(Explain on the blackboard)

Wave equation

$$u_{tt} - c^2\ u_{xx} = 0$$
$$u(x,0) = u_0(x)$$

Show video

Intuition behind:
in some sense we can "decompose" the wave equation into two transport equations "$u_t - cu_x$" and "$u_t + cu_x$"
We will see how to make a mathematically rigorous understanding of this in the future.

Exercise 1:

a) Using change of variables $\xi = x - ct$ and $\eta = x + ct$, get a simplified equation on $v(\xi, \eta) = u(x,t)$.
b) Using item a) show that there exist functions f and g such that

$$u(x,t) = f(x - ct) + g(x + ct),$$

So this means that the solution is a sum of two travelling waves moving with opposite speeds c and $-c$.

Remark:
Notice that adding two solutions of the wave equations will again be a solution (due to the linearity of the equation). This fact can be interpreted as "no interaction" of the waves. It will be not the case for the NON-linear equations (and is one of the sources of difficulty for mathematical analysis)

Next time we will discuss the wave equation in all mathematical detail.

Conservation (and balance) laws [1]

$$u_t + (f(u))_x = 0$$
$$u(x,0) = u_0(x)$$

- $u = u(x,t)$ – the conserved quantity
- $f(u)$ – the flux of conserved quantity
- $x \in \mathbb{R}$, $t \in \mathbb{R}_+$

- This formula, indeed, means “conservation”: if we take two points $x = a$ and $x = b$, then the change of total mass of u between a and b is equal to $f\big(u(a,t)\big) - f\big(u(b,t)\big)$ = [inflow at a] – [outflow at b]

- If the right-hand side is not zero (some function f, that plays a role of some “source” of mass), then this equations is called a balance law

- In problems of physics this equation is usually used to describe conservation of mass, momentum, energy etc
- This is the simplest model for water-oil displacement (the so-called Buckley-Leverett equation)
- If fact, no matter what is conserved: could be density in a crowd of people, cars, insects etc.

[1] A guru in the field C. Dafermos gives a very intuitively clear explanation how PDE for a balance law $div(F) = P$ appears from first principles. For those who are interested I advice to see Lecture 1 of his course at IMPA in 2013 (see links at the end of the slides).

Conservation (and balance) laws [1]: Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(x,0) = u_0(x)$$

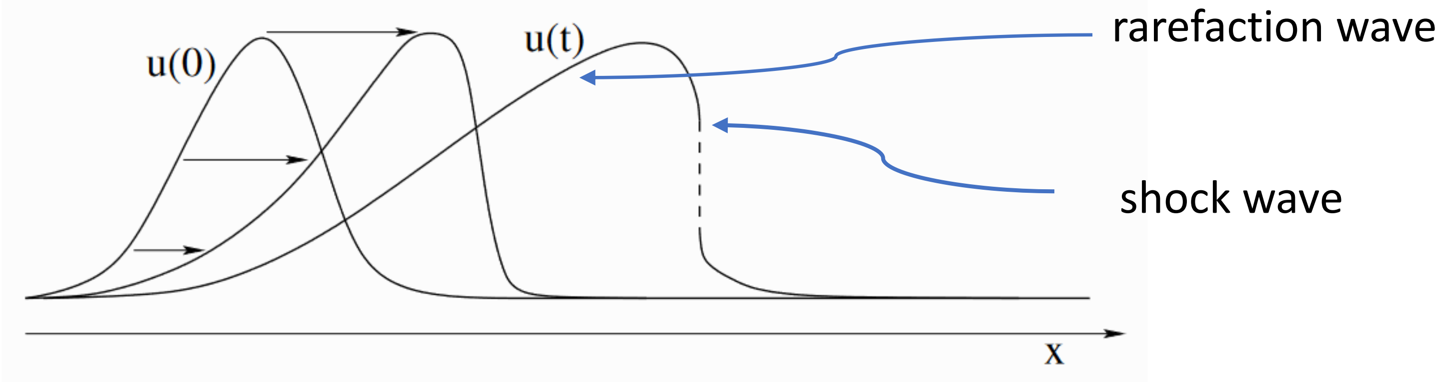
Show video with shock wave

Show video with rarefaction wave

Assume u is smooth and differentiate u^2:

$$u_t + u \cdot u_x = 0$$

"speed"



Exercise 2:
Calculate mathematically the time of "blow-up" of the solution for

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

[1] A guru in the field C. Dafermos gives a very intuitively clear explanation how PDE for a balance law $div(F) = P$ appears from first principles. For those who are interested I advice to see Lecture 1 of his course at IMPA in 2013 (see links at the end of the slides).

Problems that we are faced:

1. **In which sense the solution EXISTS?**

- Classical solution: "u should be as smooth as many derivatives are in the equation", thus
$$u \in C^1(\mathbb{R} \times \mathbb{R}_+)$$
but we see that solution may become even not a continuous function !!!
So we have a problem with existence of solutions.

- We need the notion of a **"weak" solution** (in the sense of distributions) – we want to consider a "wider" space.
Idea: look at the solution not as a function, but as a functional.

  Example: Dirac delta "function": $\delta_x: C(\Omega) \to \mathbb{R}$ such that for any $\varphi \in C(\Omega)$ we define $\delta_x(\varphi) = \varphi(x)$.
  We will consider this notion in detail later in the course.

2. **Is the solution UNIQUE?**

- As we will see, unfortunately, NOT. There are **MANY weak solutions** and this creates a problem.

- Fortunately, physics gives us a lot of restrictions (like second law of thermodynamics, entropy etc),
so this helps to choose **a unique physically relevant solution** (with quite a lot of effort, though).

Important class of solutions

It is always very useful to look for some special solutions with symmetries
(e.g. radially symmetric or having the symmetry of the equation)

$$u_t + (f(u))_x = 0$$
$$u(x,0) = u_0(x)$$

Notice that our equation is scale-invariant: $(x,t) \to (\alpha x, \alpha t)$ for any $\alpha > 0$ does not change the equation.
If we take the initial data scale-invariant, we can look for a self-similar solution of the form

$$u(x,t) = v\left(\frac{x}{t}\right)$$

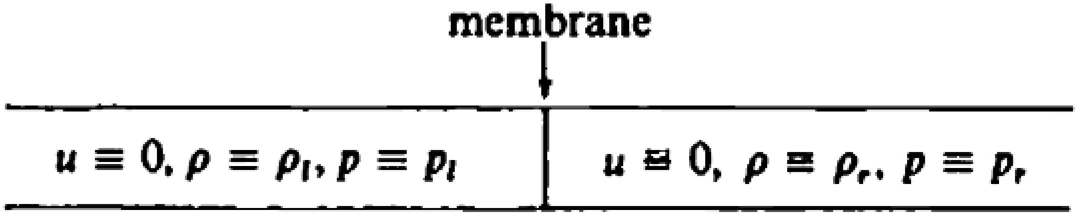
which depends on one variable, thus it satisfies some ODE (and not PDE!)

We will see how to find such solutions and why they are important:

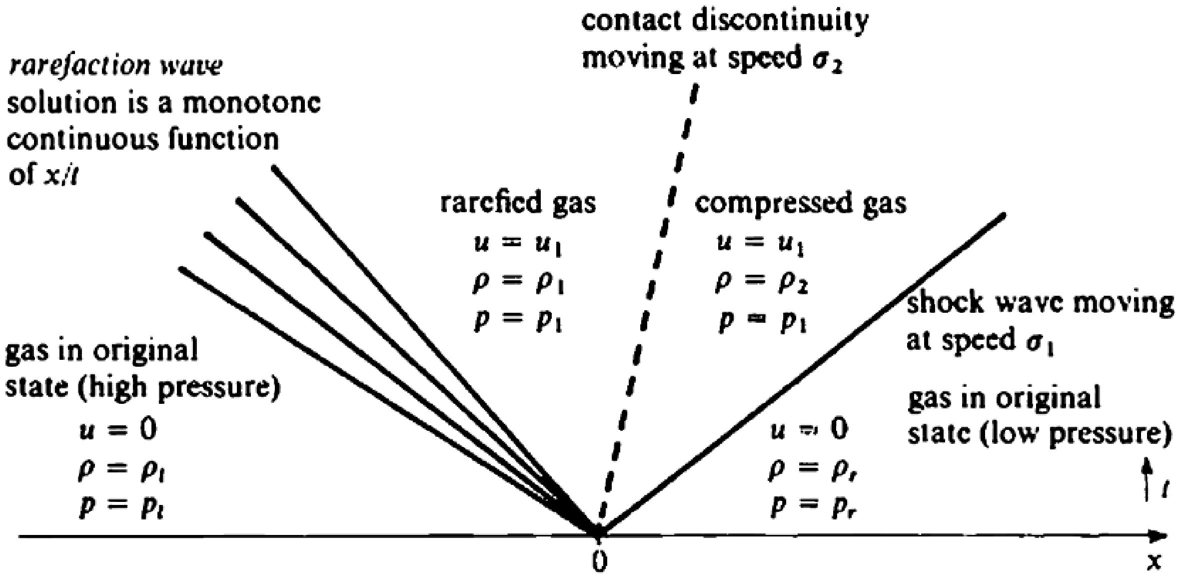
- They are building blocks for numerical scheme
- They help to prove existence of solution to a general initial data
- They appear as limiting ones when $t \to \infty$

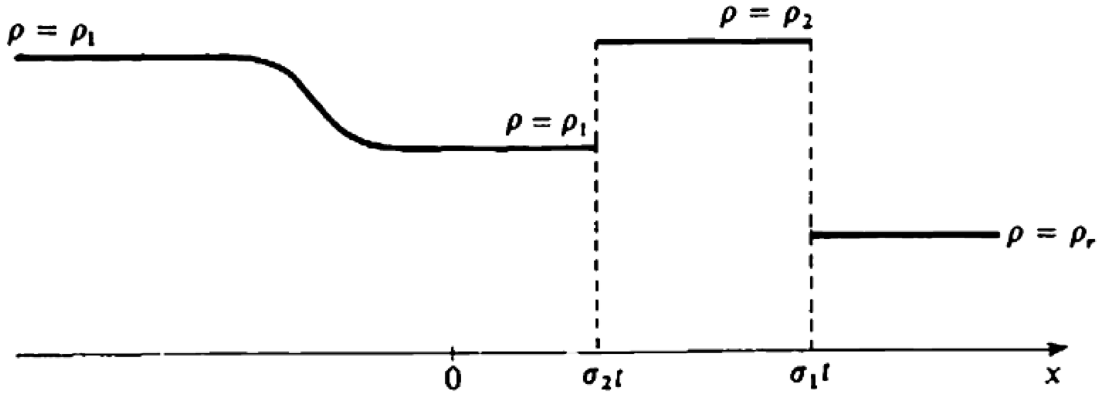
They can be rather tricky!

Riemann problem (gas dynamics)



u – velocity, ρ – density, p - pressure





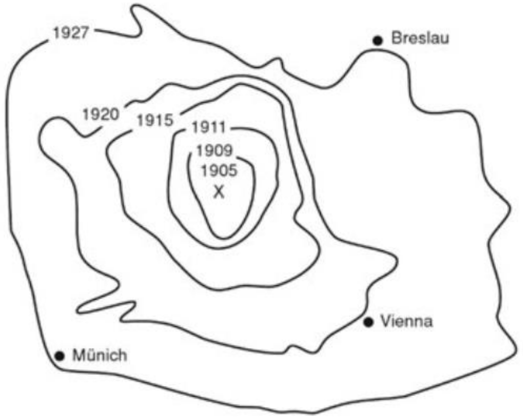
Example 4: reaction-diffusion equation

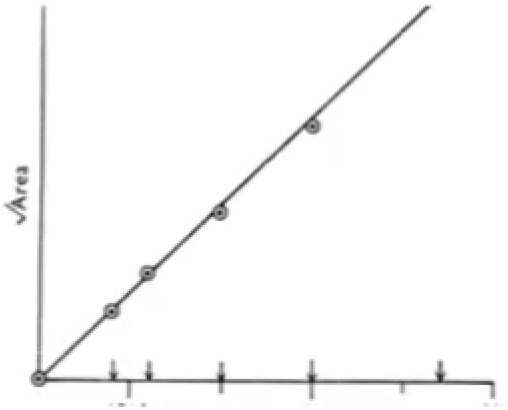
$$u_t = \underbrace{u_{xx}}_{\text{displacements}} + \underbrace{f(u)}_{\text{reproduction}}$$

This equation naturally appears in biological invasions (population dynamics), where $u = u(x,t)$ is a population density

This is a muskrat, an animal very much liked for its fur

At the beginning of the last century a few muskrats escaped from a farm in Czech republic. The result is shown below:





J.G. Skellam (1951) – describes spread of muskrats
– writes an equation like Fisher-KPP

The basic equation

Main assumptions:

1. A living population is represented by its density $u(x,t)$: number of individuals per time and space unit.
2. Individuals move and reproduce.

Variation of number of individuals at time t and place x

$=$ Number of individuals arriving at x at time t
$-$ Number of individuals leaving x at time t
$+$ Number of individuals created/annihilated at x at time t

Modelling reproduction

Ignore movements: $u(x,t) = u(t)$
Assume that reproduction rate depends only on local density

$$\dot{u}(t) = f(u)$$

1. Simplest way: $f(u) = \mu u$

2. A given piece of space can carry only a certain capacity of individuals:
   $\Rightarrow f(u)$ should be negative for large u

Simplest reproduction rate: $f(u) = \mu u \left(1 - \frac{\beta}{\mu} u\right)$

$\frac{1}{\beta}$ is called carrying capacity

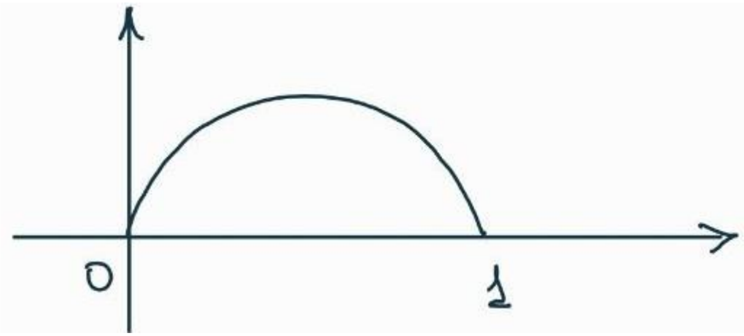
Modelling reproduction

Sometimes, population growth is limited by low densities:

- $f(u) < 0$ if u is small
- $f(u) > 0$ if u is moderately large
- $f(u) < 0$ above carrying capacity

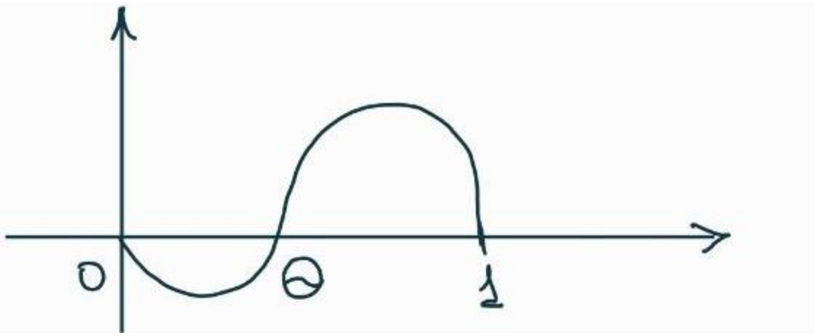
Simplest way: $f(u) = \mu u \left(1 - \frac{\beta}{\mu} u\right)(u - \theta)$

Summing up:



$f(u) = u - u^2$ (Fisher-KPP nonlinearity)

KPP = Kolmogorov, Petrovsky, Piskunov (1937)
Fisher (1930), statistician



$f(u) = u(u - \theta)(1 - u)$
(bistable nonlinearity)

Fisher-KPP

$$u_t = u_{xx} + u(1-u)$$

$$u(x,0) = \text{“gaussian hat”} \in [0,1]$$

Start to model: let’s make a “splitting”

Step 1: $u_t = u(1-u)$ pushes everything to 1

Step 2: $u_t = u_{xx}$ averages

Step 3: Repeat steps 1 and 2 sequentially

State 0 is unstable
State 1 is stable
We see an invading front! 1 invades the domain with 0.

Question: what is the speed of invasion?

Fisher KPP (first result)

Let u_0 be a Heavy-side function, that is $u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$

Theorem [Kolmogorov-Petrovsky-Piskunov, 1937]:

There exists

- a function $\sigma(t)$ such that $\frac{\sigma(t)}{t} \to 0$ as $t \to \infty$
- A function $\phi: \mathbb{R} \to \mathbb{R}$ such that
  - $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$
  - $\phi' < 0$

Such that $u(x,t)$ has a representation

$$u(x,t) = \phi\big(x - 2t + \sigma(t)\big) + o(1), \qquad t \to \infty$$

Moral: the solution behaves as a travelling wave with speed equal to 2.

Course content

1. **Linear theory:**
   a. Well-posedness and exact solution for a one-dimensional wave equation
   b. Reminder on Fourier transform
   c. The notion of weak solution (distributions, weak-derivatives, convolution, fundamental solution)

2. **Conservation and balance laws:**
   a. Definition of weak-solution
   b. Jump condition (Rankine-Hugoniot condition)
   c. Notion of hyperbolic system of conservation laws
   d. Single conservation law: existence, uniqueness, asymptotic behaviour of the entropy solution.
   e. Riemann problem: shock and rarefaction waves
   f. Entropy, Riemann invariants
   g. *(if time permits)* Vanishing viscosity method
   h. *(if time permits)* The Glimm difference scheme

3. ***(if time permits)*** **Reaction-diffusion equations:**
   a. Comparison theorems
   b. Sub- and super- solutions
   c. Speed of propagation for the Fisher-KPP equation (Aronson-Weinberger theorem)

References

Books that can be useful:

1. Evans, L.C. *Partial differential equations* (Vol. 19). American Mathematical Society.
   I advise this textbook for all who are interested in PDEs.
   Sections 3, 10, 11 are related to hyperbolic conservation laws (but not only).

2. Smoller, J. *Shock waves and reaction-diffusion equations* (Vol. 258). Springer Science & Business Media.
   My plan is to (mainly) follow this book (surely, not all the material)

3. Dafermos, C.M., 2005. *Hyperbolic conservation laws in continuum physics* (Vol. 3). Berlin: Springer.
   If you want more physics about conservation laws, this book is a good option.
   This is considered as an encyclopaedia of hyperbolic balance laws (and it is, indeed, a hard book to read).
   I advice to start with online lectures of Dafermos (see below), and if you want details on proofs see the book.

4. Bressan, A., Serre, D., Williams, M. and Zumbrun, K., 2007. *Hyperbolic systems of balance laws: lectures given at the CIME Summer School held in Cetraro, Italy, July 14-21, 2003*. Springer.

Links to online courses:

1. At IMPA in 2013 there was a mini-course of 9 lectures on "Hyperbolic conservation laws" from Constantine Dafermos. It is, indeed, very interesting, and may be I will take something from it:
   https://www.youtube.com/watch?v=WF9WrjJOLCQ&list=PLo4jXE-LdDTTg8Z4iGDNOSDA74rcwoU2a
   This is more informal interpretation of a Dafermos treatise book made by the same author.