List of exercises 5 (finally, the last one). Deadline: 16 June 2023, 23:59.

We concentrate on the maximum principle for ODEs & parabolic PDEs and its applications.
Consider second order differential operator of the form:

$$L = -\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x), \qquad x \in (a,b) \subset \mathbb{R}.$$

We suppose $u \in C^2((a,b)) \cap C([a,b])$, $g(x)$ and $h(x)$ are bounded functions.

1. (One-dimensional maximum principles for $h \not\equiv 0$)
   (a) Suppose that $h \geq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.
   (b) Suppose that $h \leq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \leq 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.
   (c) Suppose that $\max\limits_{x\in[a,b]} u(x) = M = 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.

   Hint: It is helpful to start with simpler lemma (with strict inequalities)

   Lemma 1. *Suppose that $h \geq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.*
   If $Lu < 0$, then u can attain maximum M only at the endpoints a or b.

2. (One-dimensional Hopf lemma for $h \not\equiv 0$)
   Suppose that $h \geq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.
   If $Lu \leq 0$, then:
   (a) if $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$.
   (b) if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$.

3. (Comparison theorem for semilinear parabolic equations)
   Consider a semilinear parabolic operator of the form

   $$Su := \partial_t u - \Delta u + F(t,x,u,\nabla u), \quad x \in \mathbb{R}^N, t > 0.$$

   Assume that F is C^1 jointly in all of its arguments.
   Let u be a subsolution ($Su \leq 0$) and v be a supersolution ($Sv \geq 0$).
   If $u(0,x) \leq v(0,x)$, then $u(t,x) \leq v(t,x)$.

4. (Boundedness of solution to diffusive Burgers' equation)
   Let $u \in C^2(\mathbb{R} \times (0,T]) \cap C^1(\mathbb{R} \times [0,T])$ be a solution to the one-dimensional diffusive Burgers' equation

   $$\begin{cases} \partial_t u = uu_x + u_{xx}, & \text{in } \mathbb{R} \times (0,T], \\ u = u_0, & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

   Prove that u is bounded.

In the class we mentioned the following problems. I put them here and if you are interested you can think how to solve them.

1. Consider a one-dimensional boundary value problem ($L > 0$):

   $$\begin{cases} -u'' = e^u, & x \in [0,L], \\ u(0) = u(L) = 0. \end{cases} \tag{1}$$

   Show that there exists $L_1 > 0$ such that for all $0 < L < L_1$ there exists a positive solution (in $(0,1)$) of (1), and for all $L > L_1$ there does not exist a positive solution of (1).