List of exercises 4. Deadline: 26 May 2023, 23:59.

Let us concentrate on the systems of conservation laws ($U \in \mathbb{R}^m$, $m > 1$, $F : \mathbb{R}^m \to \mathbb{R}^m$):

$$U_t + F(U)_x = 0. \tag{1}$$

1. For a fixed state $U_l \in \mathbb{R}^m$ define a shock curve (shock set or Hugoniot locus) the set of all U, such that the Rankine-Hugoniot condition is valid:

$$S(U_l) = \{U \in \mathbb{R}^m : \quad \exists \sigma = \sigma(U_l, U) \in \mathbb{R} \quad \text{such that} \quad F(U) - F(U_l) = \sigma \cdot (U - U_l)\}$$

   As we have proven the set $S(U_l)$ consists of the union of m smooth curves $S_k(U_l)$, $k = 1, \ldots, m$.

   Prove that as $U \to U_l$ and $U \in S_k(U_l)$, we have:

$$\sigma(U_l, U) = \frac{\lambda_k(U) + \lambda_k(U_l)}{2} + O(|U - U_l|^2).$$

   Here $\lambda_k(U)$ are the eigenvalues of the Jacobian matrix $DF(U)$.

   Hint: differentiate two times the Rankine–Hugonit condition at point U_l. Do the same for the expression for the eigenvalues and eigenvectors of DF:

$$DF(U) r_k(U) = \lambda_k(U) r_k(U).$$

   Combine these two equalities.

2. Let $w = (v, u)$ and let $\varphi(w)$ be a smooth scalar function. Consider the system of conservation laws

$$w_t + (\varphi(w)w)_x = 0. \tag{2}$$

   (a) Find the characteristic speeds λ_1 and λ_2 and the associated eigenvectors r_1 and r_2 for this system.

   (b) Let $\varphi(w) = |w|^2/2$. Then find the solution of the Riemann problem:

$$U(x, 0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases} \tag{3}$$