List of exercises 3. Deadline: 28 April 2023, 23:59.

1. (Irreversibility) Let the solution of the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

at $t = 1$ be equal to:

$$u(x,1) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{1}$$

Construct infinitely-many different initial conditions $u(x,0)$ (and draw them up to time $t = 1$) such that at $t = 1$ the solution coincides with (1).

2. Consider a scalar conservation law ($u \in \mathbb{R}$)

$$u_t + (f(u))_x = 0, \tag{2}$$

and the following finite-difference approximation of it:

$$\frac{u_n^{k+1} - \frac{1}{2}(u_{n+1}^k + u_{n-1}^k)}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2l} = 0. \tag{3}$$

Here $u_n^k = u(x_n, t_k)$ is defined on the grid $x_n = nl$, $t_k = kh$, $l = \Delta x > 0$, $h = \Delta t > 0$ and $l \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$. Let $u(x,0) = u_0(x)$, and $u_n^0 = u_0(x_n)$, and $M := \|u_0\|_\infty$.

Prove that:

$$|u_n^k| \leqslant M \qquad \text{for all } n \in \mathbb{Z},\ k \in \mathbb{N} \cup \{0\}.$$

3. Write a computer program, modelling (2), using an explicit finite-difference scheme defined in (3).

Show the graphs of the solution $u(\cdot, t)$ for the following Riemann problems (at several subsequent time moments):

$$1)\ u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \qquad\qquad 2)\ u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Consider two cases for the flux function f:

$$\text{a) } f(u) = 2u - u^2; \qquad\qquad \text{b) } f(u) = \frac{u^2}{u^2 + (1-u)^2}.$$

Give a theoretical explanation to the observed results in all four cases (1a, 1b, 2a, 2b).

P.S. In the implementation of the numerical scheme remember to check that the CFL (Courant-Friedrichs-Lewy) condition is fulfilled:[1]

$$\frac{A \cdot \Delta t}{\Delta x} < 1,$$

where $A = \max\limits_{u \in [0,1]} |f'(u)|$.

[1]This guarantees the convergence of the numerical scheme (3) to a solution of the original PDE (2).