List of exercises 2. Deadline: 7 April 2023, 23:59.

1. Find a Fourier series solution to the initial-boundary value problem ($t > 0$, $x \in [a,b] \subset \mathbb{R}$):

$$u_{tt} - c^2 u_{xx} = 0,$$

with initial conditions

$$u(x,0) = \varphi(x) = \begin{cases} x, & x \in [0, \pi/2] \\ \pi - x, & x \in [\pi/2, \pi] \end{cases}, \qquad u_t(x,0) = 0,$$

and boundary conditions: $u(a,t) = u(b,t) = 0$.

2. Assume that the vector field u is $\mathrm{C}_t\mathrm{Lip}_x$, and let $X(t,a)$ be a flow map, corresponding to particle trajectories under the flow of u, that is:

$$\partial_t X(t,a) = u(t, X(t,a)), \qquad X(0,a) = a \in \mathbb{R}^d.$$

Consider a flow map as a map: $a \mapsto X(t,a)$ for some fixed $t > 0$, and it's Jacobian:

$$J(t,a) := \det(\nabla_a X)(t,a) = \sum_{\varepsilon_1,\dots,\varepsilon_d=1}^{d} \varepsilon_{\varepsilon_1,\dots,\varepsilon_d} \frac{\partial X_{i_1}}{\partial a_1}(t,a) \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}(t,a),$$

where $\varepsilon_{\varepsilon_1,\dots,\varepsilon_d}$ denotes the standard Levi-Civita symbol, that is

$$\varepsilon_{\varepsilon_1,\dots,\varepsilon_d} = \begin{cases} \mathrm{sign}(\sigma), & i_n = \sigma(n) \text{ for all } n \in 1,\dots,d \text{ and some permutation } \sigma \in S_d \\ 0, & \text{otherwise.} \end{cases}$$

Prove that

$$\partial_t J(t,a) = J(t,a) \cdot \mathrm{div}(u)(t, X(t,a)).$$

3. Compute explicitly the unique entropy solution of Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = u_0(x) = \begin{cases} 1, & x < -1, \\ 0, & x \in [-1,0], \\ 2, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$.