“Shock waves in conservation laws and reaction-diffusion equations”

List of exercises 1. Deadline: 24 March 2023, 23:59.

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1. Consider a wave equation on $u(x,t)$:

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in \mathbb{R}, t \in \mathbb{R}_+.$$

Show that after the change of variables $\xi = x - ct$ and $\eta = x + ct$, the wave equation becomes

$$v_{\xi\eta} = 0,$$

where $v(\xi,\eta) = u(x,t)$. As we have shown in the lecture this immediately leads to the following general form of the solution of a wave equation (as a sum of two travelling waves moving with opposite speeds c and $-c$ and having profiles f and g, respectively):

$$u(x,t) = f(x - ct) + g(x + ct).$$

2. Consider the following initial value problem for the Burgers equation:

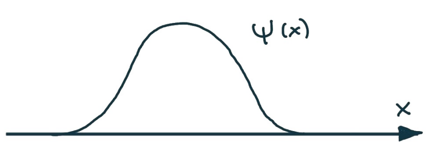
$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = u_0(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

   a) Using method of characteristics show that there exists time T, where at least two characteristic lines intersect (thus we can not define a solution u at this point). Denote by T_0 the first moment of time when some of the characteristics intersect. We will refer to such a situation as a “blow-up at time T_0”.
   b) Calculate T_0.
   c) Draw all the characteristic lines till time T_0 in the (x,t)-plane.

3. Draw a solution of the Cauchy problem for the wave equation:

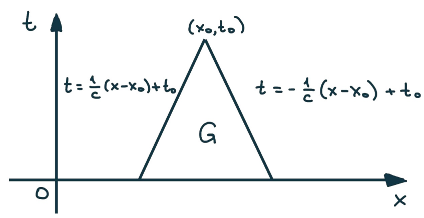
$$u_{tt} - c^2 u_{xx} = 0,$$
$$u(x,0) = \varphi(x),$$
$$u_t(x,0) = \psi(x),$$



for $\varphi \equiv 0$ and ψ depicted in figure on the right.
P.S. D’Alambert formula may help.

4. Consider a Cauchy problem for the inhomogeneous wave equation:

$$u_{tt} - c^2 u_{xx} = h(x,t).$$
$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$



Derive that the solution $u(x_0,t_0)$ takes the form:

$$u(x_0,t_0) = \frac{\varphi(x_0 - ct_0) + \varphi(x_0 + ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\,ds + \frac{1}{2c}\iint_G h(x,t)\,dxdt.$$

Here $G = \{(x,t) : t \in (0,t_0) \text{ and } x_0 + c(t - t_0) < x < x_0 - c(t - t_0)\}$ is a triangular region (see figure).
P.S. Integrate the equation over G and use the Green-Gauss theorem.