“Shock waves in conservation laws and reaction-diffusion equations”

This course was done in the Department of Mathematics at PUC-Rio during semester I, March–June 2023 by Yulia Petrova. It consists of three (in some sense) independent parts joined by the similar phenomema — the solutions of the corresponding PDEs represent some “fronts” that are propagating with time:

- Part I: wave equation (derivation, D’Alambert formula, well-posedness, Duhamel principle, solution by Fourier series)
- Part II: introduction to conservation laws (weak solution, Rankine-Hugoniot condition, entropy conditions, existence of solutions to scalar conservation law with convex flux function, exact solution to Riemann problem, existence of solutions to a strictly hyperbolic genuinely nonlinear system of conservation laws)
- Part III: introduction to reaction–diffusion equations (maximum principle for linear parabolic PDEs, comparison principle, travelling wave solutions, invasion/extinction theorems for reaction-diffusion equations with monostable and bistable nonlinearities in unbounded domains, asymptotic speed of propagation)

In this file I have collected all the materials around the course. All (possible numerous) errors are entirely mine, and I will be happy if you tell me about them through the email: yu.pe.petrova@yandex.ru.

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Useful books:

1. Smoller, J., 1983. Shock waves and reaction–diffusion equations (Vol. 258). Springer Science & Business Media.
2. Dafermos, C.M. and Dafermos, C.M., 2005. Hyperbolic conservation laws in continuum physics (Vol. 3). Berlin: Springer.
3. Evans, L.C., 1998. Partial differential equations (Vol. 19). American Mathematical Society.
4. Bressan, A.,, 2013. Hyperbolic conservation laws: an illustrated tutorial. Modelling and Optimisation of Flows on Networks: Cetraro, Italy 2009.

Useful video lectures:

1. Constantine Dafermos, course of 9 lectures at IMPA: “Hyberbolic conservation laws”
2. Henri Berestycki, mini course of 4 lectures at IMPA: “Reaction-diffusion propagation in non-homogeneous media”

1 Questions for the exam.

Part 1: Around wave equation.

1. Wave equation: "physical" derivation (balls and springs).
2. Wave equation: derivation from general principles.
3. D'Alambert's formula for 1D wave equation, and well-posedness of Cauchy problem on real line.
4. Inhomogeneous wave equation. Duhamel principle.
5. Mixed initial-boundary value problem for wave equation: existence and uniqueness of solution.
6. Mixed initial-boundary value problem for wave equation: solution by a Fourier series.

Part 2: Conservation and balance laws.

7. Fluid flow: Eulerian vs. Lagrangian point of view; flow map; incompressibility condition.
8. Fluid flow: scalar transport equation, conservation of mass.
9. Scalar conservation law. Weak form of solution. Rankine-Hugoniot condition.
10. Burgers equation: blow-up in finite time, explicit solutions to different Riemann problems, multiplicity of solutions, definition of entropy solution, irreversibility.
11. Scalar conservation law with convex flux function: various interpretations of entropy condition (Lax, Liu, vanishing viscosity).
12. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 1 and 2 describing properties for discrete approximation (boundedness, entropy condition).
13. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 3, 4 and 5 describing properties for discrete approximation (space and time estimates, stability).
14. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemma 6 on convergence and properties of the limiting solution.
15. Scalar conservation law with convex flux function: theorem on existence of entropy solution. Lemmas 7 and 8 on properties of the limiting solution.
16. Scalar conservation law with convex flux function: uniqueness of entropy solution. General plan of proof without technical details.
17. Scalar conservation law with convex flux function: uniqueness of entropy solution. Proof that $|\psi_x^m|$ is bounded using the entropy condition.
18. Scalar conservation law with convex flux function: solution to a Riemann problem for two cases ($u_l < u_r$ and $u_l > u_r$).
19. Systems of conservation laws: weak solution, Rankine–Hugoniot condition, notion of hyperbolic and strictly hyperbolic systems, examples.
20. Systems of conservation laws: notion of genuinely nonlinear and linearly degenerate characteristic family; simple waves. Theorem on existence of k-rarefaction wave.
21. Systems of conservation laws: notion of shock curves (Hugoniot locus). Theorem on structure of shock waves (property (iii) without proof). Notion of Lax admissibility criteria for shocks.
22. Systems of conservation laws: notion of k-contact discontinuity. Theorem on linear degeneracy (shock and rarefaction curves coincide). Example (linear wave equation).
23. Systems of conservation laws: theorem on local solvability of a Riemann problem for strictly hyperbolic systems (each characteristic family is genuinely nonlinear or linearly degenerate).
24. Systems of conservation laws: entropy criteria (Lax, Liu, vanishing viscosity, entropy/entropy-flux).
25. Buckley-Leverett equation (with S-shaped flux function): solution to a Riemann problem for two cases ($u_l < u_r$ and $u_l > u_r$).

Part 3: Intro to reaction-diffusion equations.

26. Reaction-diffusion equations: probabilistic justification of laplacian, examples for nonlinearities (FKPP, monostable, bistable, ignition) and their interpretation in population dynamics. Formulation of the initial-value problem.
27. Maximum principles for linear ODEs of the second order with $h \equiv 0$ (with proofs).
28. Various versions of the maximum principles for linear ODEs of the second order without the assumption that $h \equiv 0$ (with proofs). Counter-examples.
29. The idea of the "sliding method" on two examples.
30. Weak and strong maximum principle for linear parabolic PDEs for bounded domains with Dirichlet boundary conditions (with proof).
31. Weak and strong maximum principle for linear parabolic PDEs for bounded domains with Neumann/Robin boundary conditions (with proof). Hopf lemma.
32. Notions of sub- and supersolution. Comparison theorems for parabolic PDEs (with proof). Application on concrete examples.
33. Well-posedness of the scalar reaction-diffusion equations (sketch of the proof for existence, proof of uniqueness and continuous dependence on initial data).
34. Theorem on existence of traveling wave solutions to scalar reaction-diffusion equation with monostable (in particular, FKPP) nonlinearity. "Dynamical" proof (phase plane method).
35. Theorem on existence of traveling wave solutions to scalar reaction-diffusion equation with bistable nonlinearity. "Dynamical" proof (phase plane method).
36. Theorem on existence of traveling wave solutions to scalar reaction-diffusion equation with monostable nonlinearity. "PDE" proof.
37. Theorem on existence of traveling wave solutions to scalar reaction-diffusion equation with bistable nonlinearity. "PDE" proof.
38. "Hair-trigger" effect for FKPP equation (with proof).
39. Theorem on invasion for reaction-diffusion equation with bistable nonlinearity (with proof).
40. Theorem on extinction for reaction-diffusion equation with bistable nonlinearity (with proof).
41. Principle of asymptotic speed of propagation (Aronson–Wienberger theorem, with proof).

2 Exercises (homework)

2.1 List of exercises 1. Deadline: 24 March 2023, 23:59.

1. Consider a wave equation on $u(x,t)$:

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in \mathbb{R}, t \in \mathbb{R}_+.$$

Show that after the change of variables $\xi = x - ct$ and $\eta = x + ct$, the wave equation becomes

$$v_{\xi\eta} = 0,$$

where $v(\xi,\eta) = u(x,t)$. As we have shown in the lecture this immediately leads to the following general form of the solution of a wave equation (as a sum of two travelling waves moving with opposite speeds c and $-c$ and having profiles f and g, respectively):

$$u(x,t) = f(x-ct) + g(x+ct).$$

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the course in Telegram!

2. Consider the following initial value problem for the Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$
$$u(x,0) = u_0(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

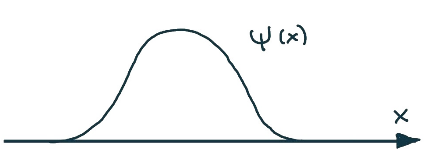
   (a) Using method of characteristics show that there exists time T, where at least two characteristic lines intersect (thus we can not define a solution u at this point). Denote by T_0 the first moment of time when some of the characteristics intersect. We will refer to such a situation as a "blow-up at time T_0".

   (b) Calculate T_0.

   (c) Draw all the characteristic lines till time T_0 in the (x,t)-plane.

3. Draw a solution of the Cauchy problem for the wave equation:

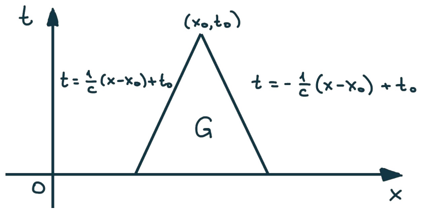
$$u_{tt} - c^2 u_{xx} = 0,$$
$$u(x,0) = \varphi(x),$$
$$u_t(x,0) = \psi(x),$$



for $\varphi \equiv 0$ and ψ depicted in figure on the right.
P.S. D'Alambert formula may help.

4. Consider a Cauchy problem for the inhomogeneous wave equation:

$$u_{tt} - c^2 u_{xx} = h(x,t).$$
$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$



Derive that the solution $u(x_0,t_0)$ takes the form:

$$u(x_0,t_0) = \frac{\varphi(x_0 - ct_0) + \varphi(x_0 + ct_0)}{2} + \frac{1}{2c}\int\limits_{x_0-ct_0}^{x_0+ct_0} \psi(s)\,ds + \frac{1}{2c}\iint\limits_G h(x,t)\,dxdt.$$

Here $G = \{(x,t) : t \in (0,t_0) \text{ and } x_0 + c(t-t_0) < x < x_0 - c(t-t_0)\}$ is a triangular region (see figure).

P.S. Integrate the equation over G and use the Green-Gauss theorem.

2.2 List of exercises 2. Deadline: 7 April 2023, 23:59.

1. Find a Fourier series solution to the initial-boundary value problem ($t > 0$, $x \in [a, b] \subset \mathbb{R}$):

$$u_{tt} - c^2 u_{xx} = 0,$$

with initial conditions

$$u(x,0) = \varphi(x) = \begin{cases} x, & x \in [0, \pi/2] \\ \pi - x, & x \in [\pi/2, \pi] \end{cases}, \qquad\qquad u_t(x,0) = 0,$$

and boundary conditions: $u(a,t) = u(b,t) = 0$.

2. Assume that the vector field u is $\mathrm{C}_t\mathrm{Lip}_x$, and let $X(t,a)$ be a flow map, corresponding to particle trajectories under the flow of u, that is:

$$\partial_t X(t,a) = u(t, X(t,a)), \qquad X(0,a) = a \in \mathbb{R}^d.$$

Consider a flow map as a map: $a \mapsto X(t,a)$ for some fixed $t > 0$, and it's Jacobian:

$$J(t,a) := \det(\nabla_a X)(t,a) = \sum_{i_1,\dots,i_d=1}^{d} \varepsilon_{i_1,\dots,i_d} \frac{\partial X_{i_1}}{\partial a_1}(t,a) \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}(t,a),$$

where $\varepsilon_{\varepsilon_1,\dots,\varepsilon_d}$ denotes the standard Levi-Civita symbol, that is

$$\varepsilon_{\varepsilon_1,\dots,\varepsilon_d} = \begin{cases} \mathrm{sign}(\sigma), & i_n = \sigma(n) \text{ for all } n \in 1,\dots,d \text{ and some permutation } \sigma \in S_d \\ 0, & \text{otherwise.} \end{cases}$$

Prove that

$$\partial_t J(t,a) = J(t,a) \cdot \mathrm{div}(u)(t, X(t,a)).$$

3. Compute explicitly the unique entropy solution of Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = u_0(x) = \begin{cases} 1, & x < -1, \\ 0, & x \in [-1,0], \\ 2, & x \in [0,1], \\ 0, & x > 1. \end{cases}$$

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$.

2.3 List of exercises 3. Deadline: 28 April 2023, 23:59.

1. (Irreversibility) Let the solution of the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

at $t = 1$ be equal to:

$$u(x, 1) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{1}$$

Construct infinitely-many different initial conditions $u(x, 0)$ (and draw them up to time $t = 1$) such that at $t = 1$ the solution coincides with (1).

2. Consider a scalar conservation law ($u \in \mathbb{R}$)

$$u_t + (f(u))_x = 0, \tag{2}$$

and the following finite-difference approximation of it:

$$\frac{u_n^{k+1} - \frac{1}{2}(u_{n+1}^k + u_{n-1}^k)}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2l} = 0. \tag{3}$$

Here $u_n^k = u(x_n, t_k)$ is defined on the grid $x_n = nl$, $t_k = kh$, $l = \Delta x > 0$, $h = \Delta t > 0$ and $l \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$. Let $u(x, 0) = u_0(x)$, and $u_n^0 = u_0(x_n)$, and $M := \|u_0\|_\infty$.

Prove that:

$$|u_n^k| \leqslant M \qquad \text{for all } n \in \mathbb{Z},\ k \in \mathbb{N} \cup \{0\}.$$

3. Write a computer program, modelling (12), using an explicit finite-difference scheme defined in (3).

Show the graphs of the solution $u(\cdot, t)$ for the following Riemann problems (at several subsequent time moments):

$$1)\ u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases} \qquad\qquad 2)\ u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Consider two cases for the flux function f:

$$\text{a) } f(u) = 2u - u^2; \qquad\qquad \text{b) } f(u) = \frac{u^2}{u^2 + (1-u)^2}.$$

Give a theoretical explanation to the observed results in all four cases (1a, 1b, 2a, 2b).

P.S. In the implementation of the numerical scheme remember to check that the CFL (Courant-Friedrichs-Lewy) condition is fulfilled:[1]

$$\frac{A \cdot \Delta t}{\Delta x} < 1,$$

where $A = \max\limits_{u \in [0,1]} |f'(u)|$.

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[1] This guarantees the convergence of the numerical scheme (3) to a solution of the original PDE (12).

2.4 List of exercises 4. Deadline: 26 May 2023, 23:59.

Let us concentrate on the systems of conservation laws ($U \in \mathbb{R}^m$, $m > 1$, $F : \mathbb{R}^m \to \mathbb{R}^m$):

$$U_t + F(U)_x = 0. \tag{4}$$

1. For a fixed state $U_l \in \mathbb{R}^m$ define a shock curve (shock set or Hugoniot locus) the set of all U, such that the Rankine-Hugoniot condition is valid:

   $$S(U_l) = \{U \in \mathbb{R}^m : \quad \exists \sigma = \sigma(U_l, U) \in \mathbb{R} \quad \text{such that} \quad F(U) - F(U_l) = \sigma \cdot (U - U_l)\}$$

   As we have proven the set $S(U_l)$ consists of the union of m smooth curves $S_k(U_l)$, $k = 1, \dots, m$.

   Prove that as $U \to U_l$ and $U \in S_k(U_l)$, we have:

   $$\sigma(U_l, U) = \frac{\lambda_k(U) + \lambda_k(U_l)}{2} + O(|U - U_l|^2).$$

   Here $\lambda_k(U)$ are the eigenvalues of the Jacobian matrix $DF(U)$.

   Hint: differentiate two times the Rankine–Hugonit condition at point U_l. Do the same for the expression for the eigenvalues and eigenvectors of DF:

   $$DF(U) r_k(U) = \lambda_k(U) r_k(U).$$

   Combine these two equalities.

2. Let $w = (v, u)$ and let $\varphi(w)$ be a smooth scalar function. Consider the system of conservation laws

   $$w_t + (\varphi(w) w)_x = 0. \tag{5}$$

   (a) Find the characteristic speeds λ_1 and λ_2 and the associated eigenvectors r_1 and r_2 for this system.

   (b) Let $\varphi(w) = |w|^2/2$. Then find the solution of the Riemann problem:

   $$U(x, 0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases} \tag{6}$$

2.5 List of exercises 5. Deadline: 16 June 2023, 23:59.

We concentrate on the maximum principle for ODEs & parabolic PDEs and its applications.
Consider second order differential operator of the form:

$$L = -\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x), \qquad x \in (a,b) \subset \mathbb{R}.$$

We suppose $u \in C^2((a,b)) \cap C([a,b])$, $g(x)$ and $h(x)$ are bounded functions.

1. (One-dimensional maximum principles for $h \not\equiv 0$)
   (a) Suppose that $h \geq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.
   (b) Suppose that $h \leq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \leq 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.
   (c) Suppose that $\max\limits_{x\in[a,b]} u(x) = M = 0$.
   If $Lu \leq 0$, then u can attain maximum M at some interior point $c \in (a,b)$ only if $u \equiv M$.

   Hint: It is helpful to start with simpler lemma (with strict inequalities)

   Lemma 1. *Suppose that* $h \geq 0$ *and* $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.
   If $Lu < 0$, *then* u *can attain maximum* M *only at the endpoints* a *or* b.

2. (One-dimensional Hopf lemma for $h \not\equiv 0$)
   Suppose that $h \geq 0$ and $\max\limits_{x\in[a,b]} u(x) = M \geq 0$.
   If $Lu \leq 0$, then:
   (a) if $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$.
   (b) if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$.

3. (Comparison theorem for semilinear parabolic equations)
   Consider a semilinear parabolic operator of the form

$$Su := \partial_t u - \Delta u + F(t,x,u,\nabla u), \quad x \in \mathbb{R}^N, t > 0.$$

   Assume that F is C^1 jointly in all of its arguments.
   Let u be a subsolution ($Su \leq 0$) and v be a supersolution ($Sv \geq 0$).
   If $u(0,x) \leq v(0,x)$, then $u(t,x) \leq v(t,x)$.

4. (Boundedness of solution to diffusive Burgers' equation)
   Let $u \in C^2(\mathbb{R}\times(0,T]) \cap C^1(\mathbb{R}\times[0,T])$ be a solution to the one-dimensional diffusive Burgers' equation

$$\begin{cases} \partial_t u = uu_x + u_{xx}, & \text{in } \mathbb{R}\times(0,T], \\ u = u_0, & \text{on } \mathbb{R}\times\{0\}. \end{cases}$$

   Prove that u is bounded.

In the class we mentioned the following problems. I put them here and if you are interested you can think how to solve them.

1. Consider a one-dimensional boundary value problem ($L > 0$):

$$\begin{cases} -u'' = e^u, & x \in [0,L], \\ u(0) = u(L) = 0. \end{cases} \tag{7}$$

   Show that there exists $L_1 > 0$ such that for all $0 < L < L_1$ there exists a positive solution (in $(0,1)$) of (7), and for all $L > L_1$ there does not exist a positive solution of (7).

3 Problem solving classes

3.1 Exercise session №1, 4 April 2023.

In this session let us concentrate on the Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{8}$$

with different initial (or boundary) conditions.

Definition 1. *A shock-wave solution, connecting states u_L and u_R and moving with speed c, is the solution of the form (for some constant states u_L and u_R):*

$$u(x,t) = \begin{cases} u_L, & x < ct, \\ u_R, & x > ct, \end{cases}$$

For a general single conservation law $u_t + (f(u))_x = 0$ there is a relation between u_L, u_R and c:

$$\text{(Rankine-Hugoniot condition = RH)} \qquad c = \frac{f(u_L) - f(u_R)}{u_L - u_R}. \tag{9}$$

1. Construct a shock-wave solution to the Burgers equation with the following conditions

$$u(x,t) = \begin{cases} 1, & x = 0, \\ 0, & t = 0, \end{cases}$$

2. Consider the Burgers equation with the following initial conditions:

$$\text{(Riemann problem)} \qquad u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

   Construct:

   (a) a smooth self-similar solution of the form: $u = v\left(\frac{x}{t}\right)$;

   (b) a shock-wave solution.

   So we have at least two solutions! Which one is “correct”?

3. Construct infinitely-many solutions to the following initial-value problem:

$$\text{(Riemann problem)} \qquad u(x,0) = \begin{cases} -1, & x < 0, \\ +1, & x > 0. \end{cases}$$

   Remark 1. *A natural question to ask is what EXTRA condition do we need to choose one solution? Such condition is usually called an “entropy” condition. An example of such condition is as follows: there exists a constant $E \in \mathbb{R}$ (independent of x, t and a):*

$$\frac{u(x+a,t) - u(x,t)}{a} \le \frac{E}{t}, \qquad a > 0, \quad t > 0. \tag{10}$$

   This condition implies that if we fix $t > 0$ and let x go from $-\infty$ to $+\infty$, then we can only jump down. Let us call the solutions that satisfy condition (10) the “entropy” solutions.

4. Which of the solutions from exercises 1–3 are entropy solutions?

5. Construct an entropy solution to the Burgers equation with the following initial conditions

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x \in [0,1], \\ 0, & x > 1, \end{cases}$$

   Consider two cases: $t \in [0,2]$ and $t \ge 2$.

6. (Irreversibility) Let the solution at $t = 1$ be equal to:

$$u(x,1) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases} \tag{11}$$

   Construct infinitely-many different initial conditions $u(x,0)$ (and draw them up to time $t = 1$) such that at $t = 1$ the solution coincides with (11).

3.2 Exercise session №2, 5 May 2023.

In this session let us concentrate on the systems of conservation laws ($U \in \mathbb{R}^m$, $m > 1$, $F : \mathbb{R}^m \to \mathbb{R}^m$):

$$U_t + F(U)_x = 0, \tag{12}$$

with Riemann initial data ($U_l, U_r \in \mathbb{R}^m$ — fixed):

$$U(x,0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases} \tag{13}$$

1. Consider a linear wave equation $w_{tt} - c^2 w_{xx} = 0$.

   It can be rewritten in the form (12) for $U = \begin{pmatrix} w_x & w_t \end{pmatrix}^T$ as follows:

   $$U_t + AU_x = 0, \qquad U = \begin{pmatrix} v \\ u \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

   (a) Find eigenvalues and eigenvectors of A;

   (b) Show that for $c \neq 0$ the system is strictly hyperbolic;

   (c) Show that the system is linearly degenerate;

   (d) Find explicit solution to (global) Riemann problem (13) for any $U_l, U_r \in \mathbb{R}^m$.

2. Consider a nonlinear wave equation $w_{tt} - (p(w_x))_x = 0$ with $p' < 0$, $p'' > 0$.

   This model comes from gas dynamics, where p is the pressure and typically $p(w) = w^{-\gamma}$ for $\gamma \geq 1$.

   It can be rewritten in the form (12) for $U = \begin{pmatrix} w_x & w_t \end{pmatrix}^T$ as follows:

   $$U_t + F(U)_x = 0, \qquad U = \begin{pmatrix} v \\ u \end{pmatrix}, \qquad F(U) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}, \qquad DF(U) = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$

   (a) Find eigenvalues and eigenvectors of $DF(U)$;

   (b) Show that if $p' \neq 0$ the system is strictly hyperbolic;

   (c) Show that if $p'' \neq 0$ the system is genuinely nonlinear for each characteristic family;

   (d) For fixed U_l find explicitly shock curves. Which part of them correspond to admissible shock waves (according to Lax admissibility criterion)? Draw 1-shock and 2-shock curves in (v,u)-plane. Draw 1-shock and 2-shock waves in (x,t)-plane.

   (e) For fixed U_l find explicitly rarefaction curves. Which part of them correspond to rarefaction waves? Draw 1-rarefaction and 2-rarefaction curves in (v,u)-plane. Draw 1-rarefaction and 2-rarefaction waves in (x,t)-plane.

   (f) Show that shock and rarefaction curves from items (d) and (e) divide the neighbourhood of U_l into 4 regions. Draw the solution to a (local) Riemann problem in (x,t)-plane considering U_r lies in one of these 4 regions.

   (g*) Show that if

   $$\int_{v_l}^{\infty} \sqrt{-p'(y)}\, dy = \infty,$$

   then there exists a solution to a global Riemann problem, that is for any U_l and U_r (not necessarily sufficiently close to each other). Is it unique?

3.3 Exercise session №3, 19 May 2023.

In this session let's concentrate on the applications of the maximum principle for ODEs with second order differential operators of the form:

$$L = -\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x), \qquad x \in (a,b) \subset \mathbb{R}.$$

Here $g(x)$ and $h(x)$ are bounded functions.

Theorem 1 (maximum principle)**.** *Let $h \equiv 0$ and $Lu \leq 0$. Then if there exists $c \in (a,b)$ such that $u(c) = \max u(x)$ for $x \in [a,b]$, then $u \equiv \max u(x)$.*

1. Does the differential operator L defined on the interval $[a,b] \subset \mathbb{R}$ provide a maximum principle? That is: if for $u \in C^2[a,b] \cap C^0(a,b)$ we have $Lu \leq 0$, then maximum of u on $[a,b]$ is obtained on the boundary (either at $x = a$ or at $x = b$).

$$\text{a)} \quad L = -\frac{d^2}{dx^2} - 1; \qquad\qquad \text{b)} \quad L = -\frac{d^2}{dx^2} + 1.$$

An important application of (different forms of) maximum principle is the following comparison principle for sub and supersolutions.

Theorem 2. *Let $f \in C^1((a,b) \times \mathbb{R})$, and let u be a subsolution and v be a supersolution, that is:*

$$Lu \leq f(x,u); \qquad\qquad Lv \geq f(x,u).$$

Then if $u(x) \leq v(x)$ for all $x \in [a,b]$, and there exists $c \in (a,b)$ such that $u(c) = v(c)$, then $u \equiv v$.

In other words, a sub-solution and a super-solution can not touch at a point: either $u \equiv v$ or $u < v$. This "untouchability" of a sub-solution and a super-solution is very helpful. See problems 2 and 3 (and the so-called sliding method).

2. Consider the boundary-value problem:

$$\begin{cases} -u'' = e^u, & 0 < x < L, \\ u(0) = u(L) = 0. \end{cases} \tag{14}$$

Prove that if L is sufficiently big, there does not exist a non-negative solution to this boundary-value problem. Proceed by the following steps:

   (a) Write a problem in terms of function $w = u + \varepsilon$:

$$\begin{cases} -w'' = e^{-\varepsilon}e^w, & 0 < x < L, \\ w(0) = \varepsilon, \quad w(L) = \varepsilon. \end{cases} \tag{15}$$

   (b) Show that functions $v_\lambda(x) = \lambda \sin(\pi x/L)$ satisfy

$$\begin{cases} -v_\lambda'' = \frac{\pi^2}{L^2} v_\lambda, & 0 < x < L, \\ v_\lambda(0) = 0, \quad v_\lambda(L) = 0. \end{cases} \tag{16}$$

   (c) Show that for big enough $L > 0$ and small enough $\lambda > 0$ the solution w of the problem (15) is a supersolution of the problem (16).

   (d) *(Sliding method)* Start increasing $\lambda > 0$ and consider the first value λ_0 such that the graphs of w and v_λ touch each other. Come to a contradiction.

   (e*) Show that there exists $L_1 > 0$ so that non-negative solution of problem (14) exists for all $0 < L < L_1$ and does not exist for all $L > L_1$.

3. Using sliding method from the previous exercise, prove that the solution u of the boundary value problem:

$$\begin{cases} -u'' - cu' = f(u), & -L < x < L, \\ u(-L) = 1, \quad u(L) = 0. \end{cases}$$

is unique.

Read more material about different kinds of maximum principle on the web-page on Miles Wheeler — Course "Theory of Partial Differential Equations"

3.4 Exercise session №4, 20 May 2023.

In this session let's concentrate on the applications of the maximum principle for linear parabolic equations:

$$\partial_t u = \Delta u + b \cdot \nabla u + cu, \qquad x \in \Omega \subset \mathbb{R}^N, t > 0. \tag{17}$$

Here $b = b(t, x)$ and $c = c(t, x)$ are continuous bounded functions. The domain Ω is either a bounded connected open set or $\mathbb{R}^N$. Using the maximum principle, we obtained the comparison principle for the semilinear parabolic equations, e.g. reaction-diffusion equations ($f \in C^1$ in u):

$$\partial_t u = \Delta u + f(t, x, u). \tag{18}$$

Theorem 3 (Weak maximum principle). *Let u be a subsolution of (17). If $u(0, x) \leq 0$, then $u(t, x) \leq 0$ for $t > 0$.*

Theorem 4 (Weak comparison principle). *Let u be a subsolution of (18) and v be a supersolution of (18). If $u(0, x) \leq v(0, x)$, then $u(t, x) \leq v(t, x)$ for $t > 0$.*

Here are some problems to solve using these theorems:

1. (Uniqueness for semilinear problems)
   Let $\Omega \subset \mathbb{R}^N$ be bounded, $f \in C^1(\mathbb{R})$, $u_0 \in C^0(\bar{\Omega})$. Prove that the problem

   $$\begin{cases} \partial_t u = -\Delta u + f(u), & \text{in } D = \Omega \times (0, T], \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } \partial\Omega \times (0, T], \end{cases}$$

   has at most one solution $u \in C^2(D) \cap C^1(\bar{D})$.

2. (Upper bound on solution for linear problems)
   Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $u(t, x)$ be the solution of the initial boundary value problem

   $$\begin{cases} u_t = \Delta u + b \cdot \nabla u + c(x)u, & \text{in } \Omega \times (0, +\infty), \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } x \in \partial\Omega \times (0, +\infty). \end{cases}$$

   Assume that the function $c(x)$ is bounded, with $c(x) \leq M$ for all $x \in \Omega$. Prove that $u(t, x)$ satisfies

   $$|u(t, x)| \leq \|u_0\|_{L_\infty} e^{Mt}, \qquad \text{for all } t > 0 \text{ and } x \in \Omega.$$

3. (Global solution vs. blow-up for reaction-diffusion equations)
   Let u be a solution to the following reaction-diffusion equation

   $$\begin{cases} \partial_t u = \Delta u + u^2, & \text{in } D_T = \Omega \times (0, T], \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \times (0, T]. \end{cases}$$

   Does the solution u blow-up in finite time?

4. (Asymptotics for the heat equation)
   Let $\Omega = B_1(0) \subset \mathbb{R}^N$ and suppose $u \in C^2(\Omega \times (0, +\infty)) \cap C^0(\bar{\Omega} \times [0, +\infty))$ satisfies for some $M > 0$:

   $$\begin{cases} \partial_t u = \Delta u, & \text{in } \Omega \times (0, +\infty), \\ |u| \leq M, & \text{on } \Omega \times \{0\}, \\ u = 0, & \text{on } x \in \partial\Omega \times (0, +\infty). \end{cases}$$

   Prove that $u(x, t) \to 0$ as $t \to \infty$ uniformly in x.

   Hint: combine the functions $2 - |x|^2$ and e^{nt} and construct a supersolution to the heat equation with appropriate behavior at $+\infty$.

4 Lecture notes

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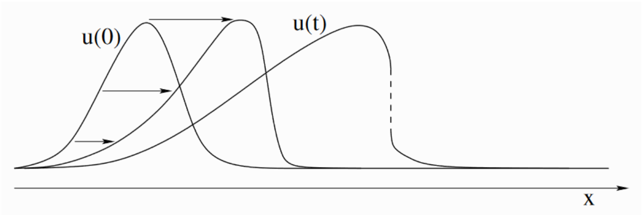
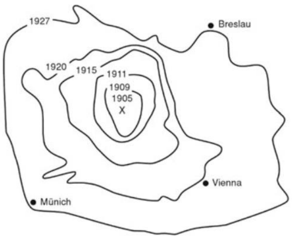
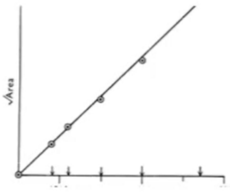
Shock waves in conservation laws and reaction-diffusion equations







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Any questions are very welcome!
Group in Telegram: https://t.me/+gdqus0TfJQ44NWI6

Yulia Petrova

Curso PUC-Rio
Rio de Janeiro, Brazil

March – June 2023

Motivation

Many phenomena in "nature" can be described using mathematical tools:

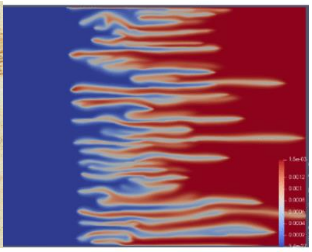
1. Physics (classical):
   - Mechanics, thermodynamics, fluid dynamics, electrodynamics
2. Biology and social sciences:
   - Population dynamics
     – how animals / bacteria / viruses / tumours spread?
   - Pattern formation
     – why do lizards have such a skin?
     – why do birds fly forming a triangle?

Basic idea:
- Create a mathematical "model"
- Study the properties of its "solutions"

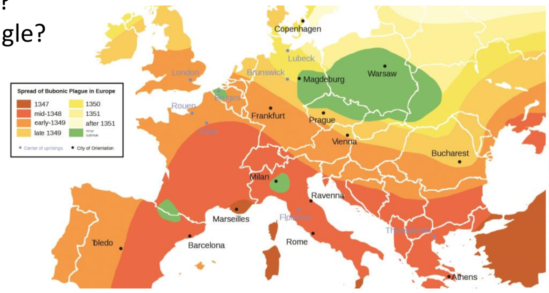
One of the conventional tools is:
PDE (partial differential equation)

Not the only one!
Probability, algebraic geometry etc…

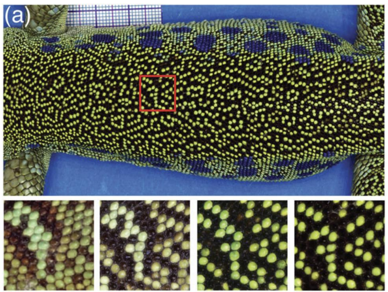
Leonardo Da Vinci describes turbulent motion of water (around 1500)



Oil recovery: displacement of oil by water



Spread of Bubonic Plague in Europe (around 1350)



29 weeks 48 weeks 61 weeks 162 weeks

What is a PDE? First example: $\Delta T = 0$

Let $T(x,t)$ be a temperature in the classroom. Here $x \in \Omega \subset \mathbb{R}^3$, $t \in \mathbb{R}_+$.

- In equilibrium:

$$\int_{\partial V} \vec{F} \cdot \nu \, dS = 0$$

$\vec{F}$ - heat flux.

- Use Green-Gauss theorem:

$$\int_{\partial V} \vec{F} \cdot \nu \, dS = \int_{V} div(\vec{F}) \, dx$$

- As this is true for all domains V, we get $div(\vec{F}) = 0.$
- Assume heat flux is proportional to gradient of temperature:

$$\vec{F} = -a \, \nabla T$$

(the more is the difference of the temperature between points, the faster is the heat flow)

Finally, we get:

$$div(\nabla T) = \Delta T = 0 \qquad \text{(Laplace equation)}$$

Pierre-Simon Laplace
(1749 – 1827)

What do you need to set up a PDE problem?

(1) Fix a domain $x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$ and consider an equation for an unknown function $u = u(x)$ for $x \in \Omega$:

$$P\left(x, u(x), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}}\right) = 0$$

The order of the highest derivative $k \in \mathbb{N}$ is called the order of the PDE.
If $n = 1$, then it is called ODE (ordinary differential equation), otherwise PDE.

(2) Fix additional boundary or initial conditions on (possibly a part) of $\partial\Omega$.

Caution: for ODEs one “typically” considers the so-called Cauchy problem:

$$u'' = f\big(t, u(t)\big)$$
$$u(0) = u$$
$$u'(0) = v$$

For PDEs the situation is more tricky and more elaborate conditions often should be considered.

Augustin-Louis Cauchy
(1789-1857)

(3) Fix to which functional space the function u belongs.

It may be $C(\Omega), C^k(\Omega)$ or some weaker spaces like $L_2(\Omega)$, Sobolev space or BV functions (functions of bounded variation)
Another thing could be that one assumes different smoothness requirements for different variables (e.g. if one of the variables corresponds to time)

Typical questions of mathematical interest:

(1) Well-posedness (in Hadamard sense, around 1902)

a. The solution exists (∃)
b. The solution is unique (!)
c. There is a continuous dependence of the solution on the "initial"/"boundary" data

- Ill-posed problems – we will see in a course

Jacques Hadamard
(1865 – 1963)

(2) Qualitative properties of the solution:

– How does the solution look like?
– Does there exist a solution of special type? E.g. having some symmetries.

- If the problem is evolutionary (there is a time variable), then a natural question is:
  – What is a long-time behaviour of the solution as $t \to \infty$?

Remark:
from my experience working with engineers the questions of existence and uniqueness are not so important for them, but the continuous dependence, indeed, is important. The reason is that there is also some noise (in the measurements, modelling etc), so it can cause big problems for them if the small change in initial data lead to big changes in solution.

A ZOO of PDEs (see Evans book on PDEs for more examples)

Linear PDEs:

(Laplace equation) $\Delta u = 0$

(Heat equation) $u_t = \Delta u$

(Linear transport equation) $u_t + \sum_{n=1}^{k} c_i u_{x_i} = 0$

(Schrodinger equation) $iu_t + \Delta u = 0$

(Wave equation) $u_{tt} - \Delta u = 0$

And many more….

Non-linear PDEs (and systems):

(Inviscid Burgers equation) $u_t + \left(\frac{u^2}{2}\right)_x = 0$

(Scalar conservation law) $u_t + div\,(F(u)) = 0$

(Scalar reaction-diffusion equation) $u_t = \Delta u + f(u)$

(Euler equation) $u_t + (u \cdot \nabla)\, u = \nabla p$
$\nabla \cdot u = 0$

(Navier-Stokes equation) $u_t + (u \cdot \nabla)\, u - \nu \Delta u = \nabla p$
$\nabla \cdot u = 0$

And many more….

A ZOO of PDEs (see Evans book on PDEs for more examples)

Linear PDEs:

(Laplace equation)	$\Delta u = 0$
(Heat equation)	$u_t = \Delta u$
(Linear transport equation)	$u_t + \sum_{n=1}^{k} c_i u_{x_i} = 0$
(Schrodinger equation)	$iu_t + \Delta u = 0$
(Wave equation)	$u_{tt} - \Delta u = 0$

And many more….

Non-linear PDEs (and systems):

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(Navier-Stokes equation)	$u_t + (u \cdot \nabla)\, u - \nu \Delta u = \nabla p$ $\nabla \cdot u = 0$

And many more….

Typical principles from Evans book on PDEs

1. Nonlinear equations are more difficult than linear equations; and, indeed, the more the nonlinearity affects the higher derivatives, the more difficult the PDE is.
2. Higher-order PDE are more difficult than lower-order PDE
3. Systems are harder than single equations
4. PDEs entailing many independent variables are harder that PDEs entailing few independent variables
5. For most PDEs it is not possible to write out explicit formulas for solutions

None of these assertions is without important exceptions.

Four main PDEs in our course:

1. **Transport equation:**

$$u_t + c\,u_x = 0$$

2. **Wave equation:**

$$u_{tt} - c^2\ u_{xx} = 0$$

3. **Scalar conservation law:**

$$u_{\mathrm{t}} + \left(F(u)\right)_x = 0$$

4. **Reaction-diffusion equation:**

$$u_t = u_{xx} + f(u)$$

They are all different (linear/non-linear), require different mathematical tools to be analysed,

BUT

Solution to these equations exhibit a “propagation” phenomena:

there are “waves” that are moving

P.S. I write the simplified version, that is for $x \in \mathbb{R}$, $u \in \mathbb{R}$, there exist various generalisations.

Transport equation

$$u_t + c\,u_x = 0$$
$$u(x,0) = u_0(x)$$

(Explain on the blackboard)

Wave equation

$$u_{tt} - c^2\ u_{xx} = 0$$
$$u(x,0) = u_0(x)$$

Show video

Intuition behind:
in some sense we can “decompose” the wave equation into two transport equations “$u_t - cu_x$” and “$u_t + cu_x$”
We will see how to make a mathematically rigorous understanding of this in the future.

Exercise 1:

a) Using change of variables $\xi = x - ct$ and $\eta = x + ct$, get a simplified equation on $v(\xi, \eta) = u(x,t)$.

b) Using item a) show that there exist functions f and g such that

$$u(x,t) = f(x - ct) + g(x + ct),$$

So this means that the solution is a sum of two travelling waves moving with opposite speeds c and $-c$.

Remark:
Notice that adding two solutions of the wave equations will again be a solution (due to the linearity of the equation). This fact can be interpreted as “no interaction” of the waves. It will be not the case for the NON-linear equations (and is one of the sources of difficulty for mathematical analysis)

Next time we will discuss the wave equation in all mathematical detail.

Conservation (and balance) laws [1]

$$u_t + (f(u))_x = 0$$
$$u(x,0) = u_0(x)$$

- $u = u(x,t)$ – the conserved quantity
- $f(u)$ – the flux of conserved quantity
- $x \in \mathbb{R}$, $t \in \mathbb{R}_+$

- This formula, indeed, means “conservation”: if we take two points $x = a$ and $x = b$, then the change of total mass of u between a and b is equal to $f\big(u(a,t)\big) - f\big(u(b,t)\big)$ = [inflow at a] – [outflow at b]

- If the right-hand side is not zero (some function f, that plays a role of some “source” of mass), then this equations is called a balance law

- In problems of physics this equation is usually used to describe conservation of mass, momentum, energy etc
- This is the simplest model for water-oil displacement (the so-called Buckley-Leverett equation)
- If fact, no matter what is conserved: could be density in a crowd of people, cars, insects etc.

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[1] A guru in the field C. Dafermos gives a very intuitively clear explanation how PDE for a balance law $div(F) = P$ appears from first principles. For those who are interested I advice to see Lecture 1 of his course at IMPA in 2013 (see links at the end of the slides).

Conservation (and balance) laws [1]: Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$
$$u(x,0) = u_0(x)$$

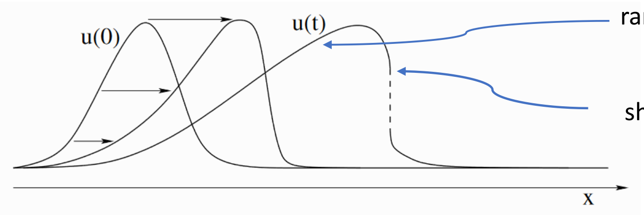
Show video with shock wave

Show video with rarefaction wave

Assume u is smooth and differentiate u^2:

$$u_t + \boxed{u} \cdot u_x = 0$$

“speed”



rarefaction wave

shock wave

Exercise 2:
Calculate mathematically the time of “blow-up” of the solution for

$$u_0(x) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

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Problems that we are faced:

1. **In which sense the solution EXISTS?**
   - Classical solution: “u should be as smooth as many derivatives are in the equation”, thus
     $$u \in C^1(\mathbb{R} \times \mathbb{R}_+)$$
     but we see that solution may become even not a continuous function !!!
     So we have a problem with existence of solutions.
   - We need the notion of a **“weak” solution** (in the sense of distributions) – we want to consider a “wider” space.
     Idea: look at the solution not as a function, but as a functional.

     Example: Dirac delta “function”: $\delta_x \colon C(\Omega) \to \mathbb{R}$ such that for any $\varphi \in C(\Omega)$ we define $\delta_x(\varphi) = \varphi(x)$.
     We will consider this notion in detail later in the course.

2. **Is the solution UNIQUE?**
   - As we will see, unfortunately, NOT. There are **MANY weak solutions** and this creates a problem.
   - Fortunately, physics gives us a lot of restrictions (like second law of thermodynamics, entropy etc), so this helps to choose **a unique physically relevant solution** (with quite a lot of effort, though).

Important class of solutions

It is always very useful to look for some special solutions with symmetries
(e.g. radially symmetric or having the symmetry of the equation)

$$u_t + (f(u))_x = 0$$
$$u(x, 0) = u_0(x)$$

Notice that our equation is scale-invariant: $(x, t) \to (\alpha x, \alpha t)$ for any $\alpha > 0$ does not change the equation.
If we take the initial data scale-invariant, we can look for a self-similar solution of the form

$$u(x, t) = v\left(\frac{x}{t}\right)$$

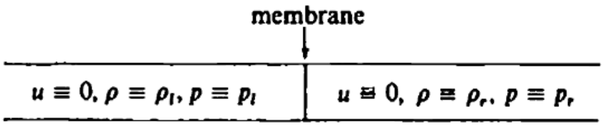
which depends on one variable, thus it satisfies some ODE (and not PDE!)

We will see how to find such solutions and why they are important:

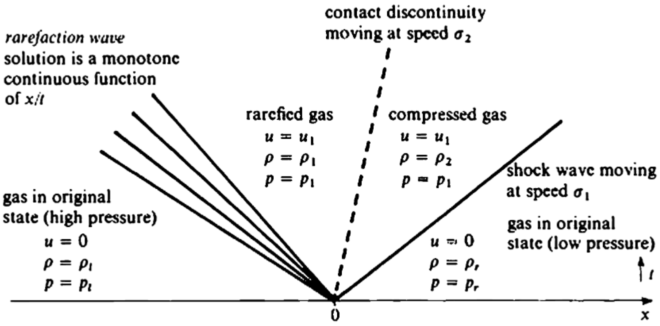
- They are building blocks for numerical scheme
- They help to prove existence of solution to a general initial data
- They appear as limiting ones when $t \to \infty$

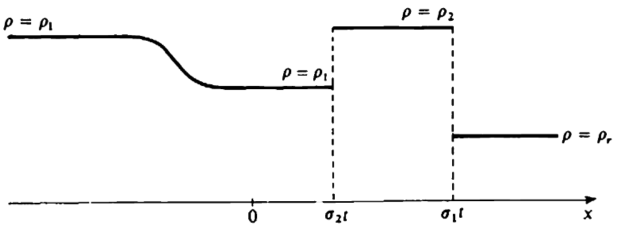
They can be rather tricky!

Riemann problem (gas dynamics)



u – velocity, ρ – density, p - pressure





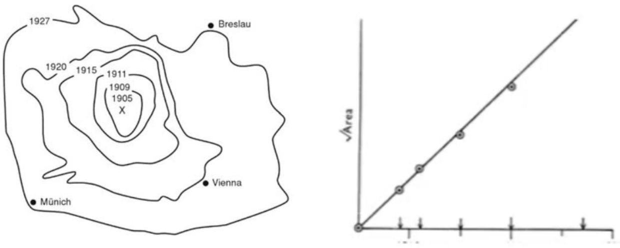
Example 4: reaction-diffusion equation

$$u_t = \underbrace{u_{xx}}_{\text{displacements}} + \underbrace{f(u)}_{\text{reproduction}}$$

This equation naturally appears in biological invasions (population dynamics), where $u = u(x,t)$ is a population density

This is a muskrat, an animal very much liked for its fur

At the beginning of the last century a few muskrats escaped from a farm in Czech republic. The result is shown below:



J.G. Skellam (1951) – describes spread of muskrats
– writes an equation like Fisher-KPP

The basic equation

Main assumptions:

1. A living population is represented by its density $u(x,t)$: number of individuals per time and space unit.
2. Individuals move and reproduce.

Variation of number of individuals at time t and place x

$=$ Number of individuals arriving at x at time t
$-$ Number of individuals leaving x at time t
$+$ Number of individuals created/annihilated at x at time t

Modelling reproduction

Ignore movements: $u(x,t) = u(t)$
Assume that reproduction rate depends only on local density

$$\dot{u}(t) = f(u)$$

1. Simplest way: $f(u) = \mu u$
2. A given piece of space can carry only a certain capacity of individuals:
   $\Rightarrow f(u)$ should be negative for large u

Simplest reproduction rate: $f(u) = \mu u \left(1 - \frac{\beta}{\mu} u\right)$

$\frac{1}{\beta}$ is called carrying capacity

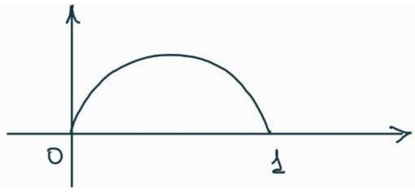
Modelling reproduction

Sometimes, population growth is limited by low densities:

- $f(u) < 0$ if u is small
- $f(u) > 0$ if u is moderately large
- $f(u) < 0$ above carrying capacity

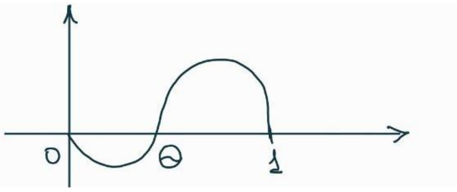
Simplest way: $f(u) = \mu u \left(1 - \frac{\beta}{\mu} u\right)(u - \theta)$

Summing up:



$f(u) = u - u^2$ (Fisher-KPP nonlinearity)

KPP = Kolmogorov, Petrovsky, Piskunov (1937)
Fisher (1930), statistician



$f(u) = u(u - \theta)(1 - u)$
(bistable nonlinearity)

Fisher-KPP

$$u_t = u_{xx} + u(1-u)$$
$$u(x,0) = \text{“gaussian hat”} \in [0,1]$$

Start to model: let’s make a “splitting”

Step 1: $u_t = u(1-u)$ pushes everything to 1

Step 2: $u_t = u_{xx}$ averages

Step 3: Repeat steps 1 and 2 sequentially

State 0 is unstable
State 1 is stable
We see an invading front! 1 invades the domain with 0.

Question: what is the speed of invasion?

Fisher KPP (first result)

Let u_0 be a Heavy-side function, that is $u_0(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$

Theorem [Kolmogorov-Petrovsky-Piskunov, 1937]:

There exists

- a function $\sigma(t)$ such that $\frac{\sigma(t)}{t} \to 0$ as $t \to \infty$
- A function $\phi: \mathbb{R} \to \mathbb{R}$ such that
  - $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$
  - $\phi' < 0$

Such that $u(x,t)$ has a representation

$$u(x,t) = \phi\big(x - 2t + \sigma(t)\big) + o(1), \qquad t \to \infty$$

Moral: the solution behaves as a travelling wave with speed equal to 2.

Course content

1. **Linear theory:**
    a. Well-posedness and exact solution for a one-dimensional wave equation
    b. Reminder on Fourier transform
    c. The notion of weak solution (distributions, weak-derivatives, convolution, fundamental solution)

2. **Conservation and balance laws:**
    a. Definition of weak-solution
    b. Jump condition (Rankine-Hugoniot condition)
    c. Notion of hyperbolic system of conservation laws
    d. Single conservation law: existence, uniqueness, asymptotic behaviour of the entropy solution.
    e. Riemann problem: shock and rarefaction waves
    f. Entropy, Riemann invariants
    g. *(if time permits)* Vanishing viscosity method
    h. *(if time permits)* The Glimm difference scheme

3. ***(if time permits)* Reaction-diffusion equations:**
    a. Comparison theorems
    b. Sub- and super- solutions
    c. Speed of propagation for the Fisher-KPP equation (Aronson-Weinberger theorem)

References

Books that can be useful:

1. Evans, L.C. *Partial differential equations* (Vol. 19). American Mathematical Society.
   I advise this textbook for all who are interested in PDEs.
   Sections 3, 10, 11 are related to hyperbolic conservation laws (but not only).

2. Smoller, J. *Shock waves and reaction-diffusion equations* (Vol. 258). Springer Science & Business Media.
   My plan is to (mainly) follow this book (surely, not all the material)

3. Dafermos, C.M., 2005. *Hyperbolic conservation laws in continuum physics* (Vol. 3). Berlin: Springer.
   If you want more physics about conservation laws, this book is a good option.
   This is considered as an encyclopaedia of hyperbolic balance laws (and it is, indeed, a hard book to read).
   I advice to start with online lectures of Dafermos (see below), and if you want details on proofs see the book.

4. Bressan, A., Serre, D., Williams, M. and Zumbrun, K., 2007. *Hyperbolic systems of balance laws: lectures given at the CIME Summer School held in Cetraro, Italy, July 14-21, 2003*. Springer.

Links to online courses:

1. At IMPA in 2013 there was a mini-course of 9 lectures on “Hyperbolic conservation laws” from Constantine Dafermos. It is, indeed, very interesting, and may be I will take something from it:
   https://www.youtube.com/watch?v=WF9WrjJOLCQ&list=PLo4jXE-LdDTTg8Z4iGDNOSDA74rcwoU2a
   This is more informal interpretation of a Dafermos treatise book made by the same author.

$\boxed{\text{Wave equation}}$

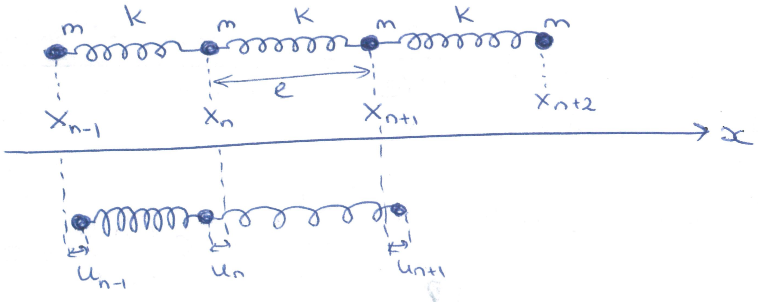
$$\frac{\partial^2 u}{\partial t^2} - \underset{c}{\cancel{a}}^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in \mathbb{R}\times\mathbb{R}_+$$

c-wave speed

Lecture 2

Plan: ① Derivation
② D'Alambert exact solution
③ Well-posedness
④ Inhomogeneous wave equation (exercise)
⑤ Mixed initial-boundary value problem by spectral method
(5a) ∃ and ! (5b) Exact solution

① Derivation 1 : (from physics)



- Position at rest: $x_n = n\ell$, $n \in \mathbb{Z}$
- k - spring constant (measure spring's stiffness)

$(\ldots, u_{n-1}, u_n, u_{n+1}, u_{n+2}, \ldots)$ - vector of horizontal displacements

Second Newton's law for the n-th mass :

$$\longrightarrow m\ddot{u}_n = F_n^- + F_n^+$$

Hooke's law : $F_n^- = -k(u_n - u_{n-1})$; $F_n^+ = k(u_{n+1} - u_n)$ where k is the elastic constant of each spring.

So we get: $m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n)$ (1)

Another way to derive is through Lagrangian :

$$L = \frac{m}{2}\sum_n \dot{u}_n^2 - \frac{k}{2}\sum_n (u_{n+1} - u_n)^2$$

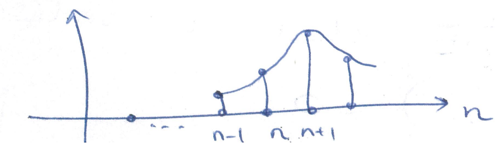
Euler-Lagrange equation : $\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_n} - \frac{\partial L}{\partial u_n} = 0$, $n \in \mathbb{Z}$

$$m\ddot{u}_n + k(u_n - u_{n-1}) - k(u_{n+1} - u_n) = 0$$

$$m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n)$$

Assume " Slow" dependence on n:

$u_n(t) \approx u(x_n, t)$ is a smooth function



$$u_{n+1} = u(x_{n+1},t) = u(x_n+\ell,t) = u(x_n,t) + \ell\frac{\partial u}{\partial x}(x_n,t) + \frac{1}{2}\ell^2\frac{\partial^2 u}{\partial x^2}(x_n,t) + O(\ell^3)$$

$$u_{n-1} = u(x_{n-1},t) = u(x_n-\ell,\ t) = u(x_n,t) - \ell\frac{\partial u}{\partial x}(x_n,t) + \frac{1}{2}\ell^2\frac{\partial^2 u}{\partial x^2}(x_n,t) + O(\ell^3)$$

$$u_n = u(x_n,t)$$

Then

$$u_{n+1} + u_{n-1} - 2u_n = \ell^2\frac{\partial^2 u}{\partial x^2}(x_n,t) + O(\ell^3)$$

$\Rightarrow$ So eq. (1) takes the form:

$$m\frac{\partial^2 u}{\partial t^2} = k\ell^2\frac{\partial^2 u}{\partial x^2} + \text{l.o.t.}$$

Denoting $\frac{k\ell^2}{m} = c^2$ $\left(\text{so } c^2 = \frac{k\ell}{m/\ell} = \frac{\text{constant of spring of length}}{\text{density per unit length}}\right.$

we will call it sound speed.

Forgetting l.o.t, we get $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$.

Wave equation is universal!

Derivation 2: [from symmetries and qualitative assymptions.]

space $x \in \mathbb{R}$, time $t \in \mathbb{R}_+$, $u(x,t) \in \mathbb{R}$
↑ state of the system

(H1) For any $u_0 \in \mathbb{R}$, the constant state $u(x) \equiv u_0$ is a stable equilibrium for any u_0

(H2) We consider small oscillations near $u(x,t) \equiv 0$

(H3) The system is homogeneous in space and time

(H4) Parity symmetry: $x \mapsto -x$ (In 3d it would corresp. left hand to right hand)

(H5) Time reversal symmetry: $t \to -t$

(H6) Long-wave approximations (oscillations are large-scale in space and time) ↑ are slowly changing in space/time

The general equation of motion for $u(x,t)$

$u(x,t) \xleftrightarrow[\text{series}]{\text{Taylor}} u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \dots$ at (x,t)

We are looking for a PDE that u satisfies

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \dots\right) = 0$$

(H3) $\Rightarrow$ $\mathcal{F}$ does not depend on (x,t)

(H2) $\Rightarrow$ we can linearize $\mathcal{F}$ at 0 $\left(u=0, \frac{\partial u}{\partial x}=0, \dots\right)$

$$C + c_{00}u + c_{10}\frac{\partial u}{\partial t} + c_{01}\frac{\partial u}{\partial x} + c_{20}\frac{\partial^2 u}{\partial t^2} + c_{11}\frac{\partial^2 u}{\partial t \partial x} + c_{02}\frac{\partial^2 u}{\partial x^2} + \dots = 0$$

$(H1)^{\overset{u_0=0}{\vee}} \Rightarrow$ $c=0$

(H1) $u_0 \not\equiv \text{const} \neq 0$, $\frac{\partial u}{\partial x}=0$, $\frac{\partial u}{\partial t}=0, \dots$ $\Rightarrow$ $c_{00}u_0=0 \Rightarrow c_{00}=0$

$(H4)^{\overset{x \mapsto -x \Rightarrow}{\vee}}$ all odd derivatives w.r.t x change sign

odd-order derivatives w.r.t.x

$\Rightarrow$ no terms with odd-order derivatives w.r.t.x

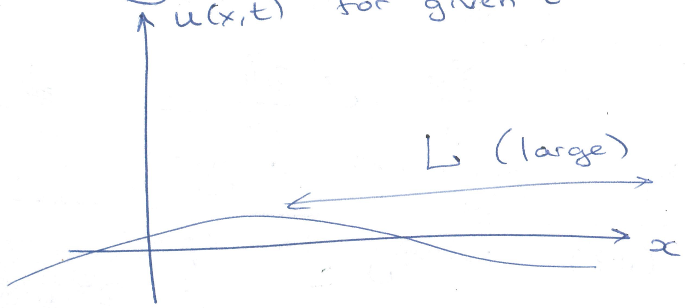
(H5) $t \to -t$ $\Rightarrow$ no terms with odd-order derivatives w.r.t. t

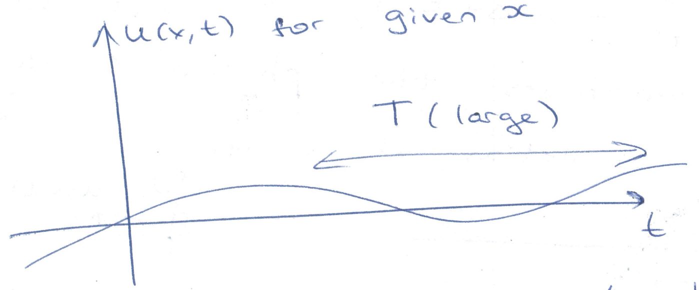
$$c_{20}\frac{\partial^2 u}{\partial t^2} + c_{02}\frac{\partial^2 u}{\partial x^2} + c_{40}\frac{\partial^4 u}{\partial t^4} + c_{22}\frac{\partial^4 u}{\partial t^2 \partial x^2} + c_{04}\frac{\partial^4 u}{\partial x^4} + \dots = 0$$

(H6) Long-wave approximation: u changes on large scales L in x

and

u changes on large scales T in t





Change variables

$\xi = \frac{x}{L}$, $\tau = \frac{t}{T}$

$\mathcal{U}(\xi,\tau) = u(x,t) =$
$= u(L\xi, T\tau)$

$\Downarrow$

$\mathcal{U}(\xi,\tau)$ changes at characteristic scales $\xi \sim 1$, $\eta \sim 1$

$$\Rightarrow \quad \frac{\partial^{2m+2n}\mathcal{U}}{\partial \xi^{2m}\partial \eta^{2n}} \sim 1 \quad \text{for all } m,n$$

$$\frac{\partial^{2m+2n}u}{\partial x^{2m}\partial t^{2n}} \sim \frac{1}{T^{2m}L^{2n}} \underbrace{\frac{\partial^{2m+2n}\mathcal{U}}{\partial \xi^{2m}\partial \eta^{2n}}}_{1} \sim \frac{1}{T^{2m}L^{2n}}$$

$x=L\xi$ $\quad$ $t=T\tau$

In long-wave approximation we conside T and L large, so

the higher-order derivative terms are small
$\Rightarrow$ we can neglect terms with higher-order derivatives

$$\Rightarrow \quad C_{20}\frac{\partial^2 u}{\partial t^2} + C_{02}\frac{\partial^2 u}{\partial x^2} = 0 \qquad \left|\frac{C_{02}}{C_{20}}\right| = ac^2$$

$$\frac{\partial^2 u}{\partial t^2} \pm c^2 \frac{\partial^2 u}{\partial x^2} = 0.$$ What is the sign ~~of~~ before c?

If $\oplus$, then $\frac{\partial^2 u}{\partial t^2} + c^2\frac{\partial^2 u}{\partial x^2} = 0.$

Then $u(x,t) = c e^{akt}\cos(kx)$ is a solution for any c and k!

$t=0$: $u(x,0) = c\cdot\cos(kx)$ if c is small.
But $u(x,t) \to \infty$ as $t \to \infty$

So you start with arbitrary small initial condition, you grow to infinity, so the $u=0$ is unstable equilibrium.

(H1) $\Rightarrow$ say $\ominus$ -sign $\Rightarrow$ $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$

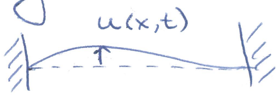
We never used any physical origin!

Observation: $x \in \mathbb{R}^3$ (H7) Isotropy in space

$\boxed{a=c}$ $\Rightarrow$ $\frac{\partial^2 u}{\partial t^2} - a^2\Delta u = 0$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

a is what characterizes the physical / biolocical ... system

Examples: ① String: oscillation of a string



$a = \sqrt{T/\rho}$, T - tension, ρ - density. u(x,t) - oscillation from an equilibrium

② Sound wave in gas or liquid
- u can be pressure or displacement of particles

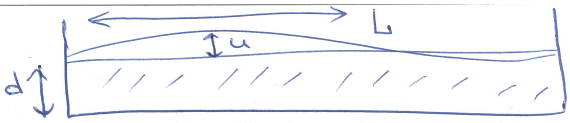
Here a is a sound speed.

③ Electromagnetic wave
- u is a field (electric or magnetic)
- a is a light speed

④ Shallow water waves

- small amplitude
- "long" waves : $H << L$

depth is much smaller than the length of wave



$c = \sqrt{gH}$; $u(x,t)$ – displacement of the equilibrium

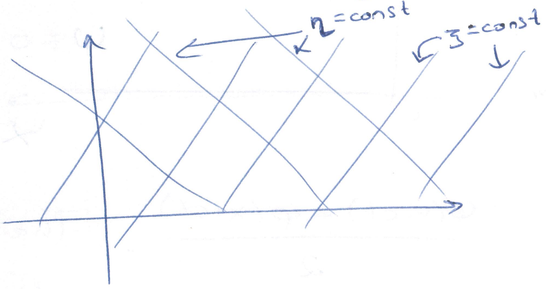
Well-posedness of Cauchy problem for 1D wave eq.

$$u_{tt} - c^2 u_{xx} = 0, \quad c \in \mathbb{R}, \ x \in \mathbb{R}, \ t \in \mathbb{R}_+$$

Let's find a general form of solution.

Make the change of variables: $\xi = x - ct$
$\eta = x + ct$

$u(x,t) = v(\xi, \eta)$



Using exercise 1, we get

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \xi}\right) = 0 \Rightarrow \frac{\partial v}{\partial \xi} = \underbrace{F(\xi)}_{\text{arbitrary}}$$

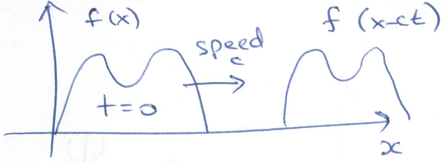
Integrate w.r.t. ξ $\Rightarrow$ $v(\xi,\eta) = \underbrace{\int F(\xi) d\xi}_{\text{arbitrary}} + \underbrace{g(\eta)}_{\text{arbitrary}}$

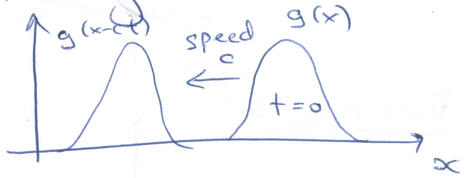
$\Rightarrow$ $v(\xi,\eta) = f(\xi) + g(\eta)$ – general solution

Thus, $\boxed{u(x,t) = f(x-ct) + g(x+ct)}$

f represents wave moving to the right

g represents wave moving to the left





Well-posedness :
1) ∃ (existence)
2) ! (uniqueness)
3) continuous dependence on initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We will prove the ∃ and ! by providing an explicit formula for solution.

$$\left.\begin{aligned} u(x,0) = \varphi(x) &\Rightarrow \quad f(x) + g(x) = \varphi(x) \\ u_t(x,0) = \psi(x) &\Rightarrow \quad -cf'(x) + cg'(x) = \psi(x) \end{aligned}\right| \Rightarrow \begin{aligned} f' + g' &= \varphi \\ -f' + g' &= \frac{\psi}{c} \end{aligned}$$

Thus, $f' = \frac{1}{2}\varphi' - \frac{1}{2c}\psi$ $\Rightarrow$ $\begin{cases} f = \frac{1}{2}\varphi - \frac{1}{2c}\int\psi(z)dz + c_1 \\ g = \frac{1}{2}\varphi + \frac{1}{2c}\int\psi(z)dz + c_2 \end{cases}$

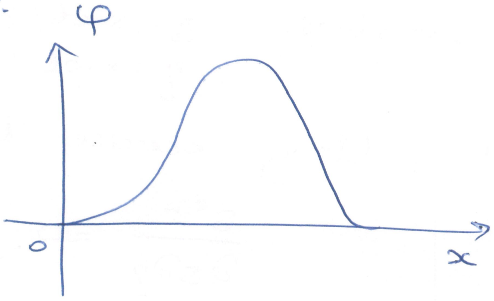
$g' = \frac{1}{2}\varphi' + \frac{1}{2c}\psi$

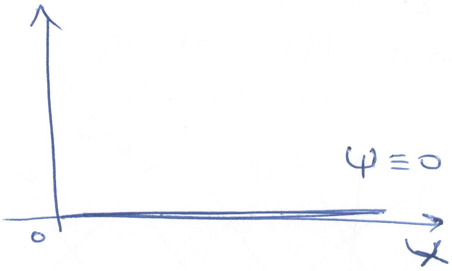
Note that $f+g=\varphi$ $\Rightarrow$ $c_1 + c_2 = 0$

Then

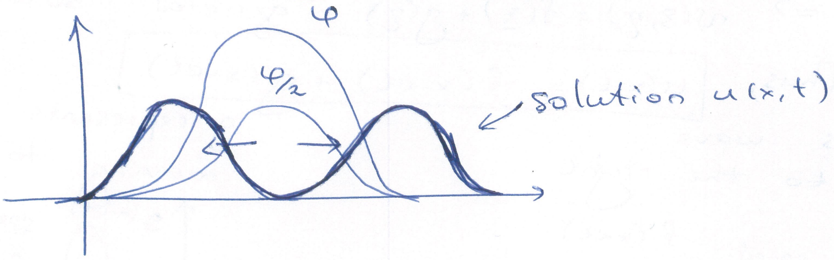
$$\boxed{u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(z)dz}$$ D'Alembert formula

Example:

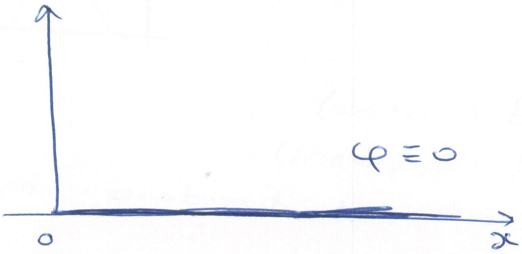


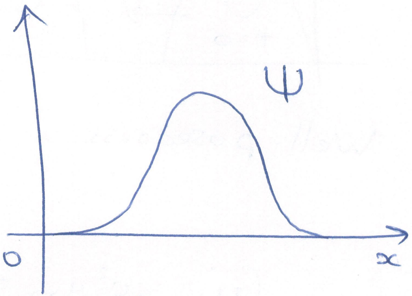


$\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv 0$. Then $u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2}$. That means we will have exactly 2 waves with profiles $\frac{\varphi(x)}{2}$ going to left and right



Exercise 3:





Draw a solution u in this case.

[redacted]

Wave equation

$$(*)\ \begin{cases} u_{tt} - c^2 u_{xx} = 0, & c \in \mathbb{R} \text{ - wave speed} \\ u(x,0) = \varphi(x) & x \in \mathbb{R},\ t>0 \\ u_t(x,0) = \psi(x) \end{cases}$$

Last time we proved that $\exists!$ solution to $(*)$.
~~To~~ And derived D'Alambert formula for $u(x,t)$:

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)\,dz.$$

To finish the proof of well-posedness, we need to show the continuous dependence on initial data.

Remarks: for classical solution we see that we "need $u(x,t) \in C^2(\mathbb{R}\times[0,+\infty))$. So it makes sense t ask that $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$.

For simplicity, let us show the continuous dependenc in the uniform norm $C(\mathbb{R})$, that is:

Lemma (cont. dependence in $C(\mathbb{R})$):

Let $\|\varphi-\varphi_1\|_{C(\mathbb{R})} < \varepsilon$ and $\|\psi-\psi_1\|_{C(\mathbb{R})} < \varepsilon$, $\varepsilon >$

and v is the solution of $(*)$ with $v(x,0)=\varphi($
$v_t(x,0)=\psi($

and v_1 is the solution of $(*)$ with $v_1(x,0)=\varphi_1($
$v_{1t}(x,0)=\psi_1($

Then, for any $T>0$ if $\varepsilon\to 0$
we have $v - v_1 \to 0$ uniformly in $x \in [0,T$

Proof:

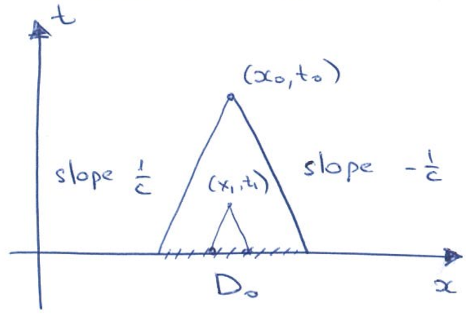
$$\|v-v_1\|_{C(\mathbb{R})} \leq \frac{\overset{<\varepsilon}{|\varphi(x-ct)-\varphi_1(x-ct)|}}{2} + \frac{\overset{<\varepsilon}{|\varphi(x+ct)-\varphi(x+ct)|}}{2}$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{<\varepsilon}{|\psi(z)-\psi_1(z)|}\,dz$$

Thus

$$\|v-v_1\|_{C(\mathbb{R})} \leq \varepsilon(1+t) \leq \varepsilon(1+T) \to 0 \quad \text{as } \varepsilon\to$$

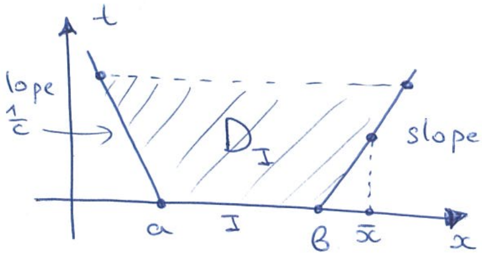
■

- **Domain of dependence** : $D_0 := \{x : x_0 - c t_0 < x < x_0 + c t_0\}$



So by D'Alambert formula we see that $u(x_0, t_0)$ depends only on values φ in points $(x_0 - c t_0)$ and $(x_0 + c t_0)$ and ψ on D_0. If we change φ and ψ outside D_0, the solution $u(x_0,$ will not change. That's why we call D_0 the domain of dependence.
Notice that for any point (x_1, t_1) inside triangle the domain of dependence is also inside ~~a triange~~ D_0.

- **Reversed question** : what points are influenced by the data in an interval I on $t=0$



$I = [a, b]$
D_I - domain of influence of I
$D_I = \{(x,t) : t \in [0, T]$ and
$a - ct \leq x \leq b + ct\}$

We say that disturbances propagate at speed c
We mean the following:
Let φ and ψ be supported on I ($\varphi = 0$, $\psi = 0$ out of I)
Imagine the observer is at point $\bar{x} \notin I$, say $\bar{x} > b$
For all times $t < \frac{\bar{x} - b}{c}$ the solution u will be 0 (the observer doesn't feel the disturbance). However, once $t \geq \frac{\bar{x} - b}{c}$ the solution will depend on φ, ψ forever!

Remark : interesting observation that we do not touch in these lectures :

think of the sound! → in $\mathbb{R}^3$ (in fact in $\mathbb{R}^{2d+1}$, $d \in \mathbb{N}$) if we hear some signal we start to hear it and finish to hear it at some point, that is the solution $\exists t_1 < t_2$ s.t.

$$u(x,t) = 0, \text{ if } t < t_1 \text{ and } u(x,t) = 0 \text{ if } t > t_2.$$

As we see in $\mathbb{R}^1$ this is not the case!
Also it is not the case for $\mathbb{R}^{2d}$, $d \in \mathbb{N}$.

Inhomogeneous wave equation:

$$(**)\quad \begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \qquad \begin{matrix} h \in C(\mathbb{R} \times [0,+\infty)) \\ \varphi \in C^2 \\ \psi \in C^1 \end{matrix}$$

→ Use linearity: consider

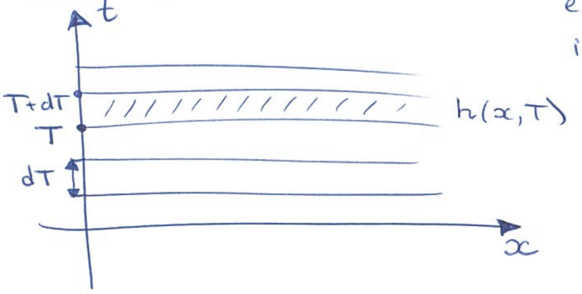
$$1)\ \begin{cases} (u_1)_{tt} - c^2 (u_1)_{xx} = h(x,t) \\ u_1(x,0) = 0 \\ (u_1)_t(x,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (u_2)_{tt} - c^2 (u_2)_{xx} = 0 \\ u_2(x,0) = \varphi(x) \\ (u_2)_t(x,0) = \psi(x) \end{cases} \ (2$$

then $u(x,t) = u_1(x,t) + u_2(x,t)$

We know how to solve (2). How to solve (1)? Let me give you "a general construction" that allo to solve (1) if you know how to solve (2). It is called Duhamel principle.

The idea is to move $h(x,t)$ from RHS (right hand sid of (1)) to the initial data in (2).

Intuition (physical): the term $h(x,t)$ acts as an external force at every point x in space and time t.



Let's divide the (x,t)-place into strips of infinitesimal lengths dT, and assume the forcing there is consta

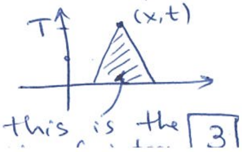
$$h(x,T) \Rightarrow u_{tt} \sim h(x,T)$$
$$u_t \sim h(x,T)\,dT$$

So we can consider an auxiliary problem:

$$\begin{cases} \mathcal{U}_{tt} - c^2 \mathcal{U}_{xx} = 0 \\ \mathcal{U}(x,T) = 0 \\ \mathcal{U}_t(x,T) = h(x,T)\,dT \end{cases} \underset{\text{D'Alamb.}}{\Rightarrow} \mathcal{U}(x,t) = \frac{1}{2c}\int_{x-c(t-T)}^{x+c(t-T)} h(\cancel{x}z,T)\,dz \cdot d$$

If we now consider the time from $t=0$ to $t=T+d$ we need to sum up all $\frac{1}{2c}\int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz \cdot dT$, that is to take integral:

$$\frac{1}{2c}\int_0^t \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz\,dT$$



this is the

Let's prove this mathematically rigorous.

Duhamel principle: take $v = v(x, t; s)$ such that

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 &, t > s \\ v(x,t;s) = 0 &, t = s \\ v_t(x,t;s) = f(x,s) &, t = s \end{cases}$$

Then $u(x,t) = \int_0^t v(x,t;s)\,ds$ is a solution to (1)

Proof:

⌜▶ $u_t = \cancelto{0}{v(x,t;t)} + \int_0^t v_t(x,t;s)\,ds$

$$u_{tt} = v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,ds =$$

$$= f(x,t) + \int_0^t v_{tt}(x,t;s)\,ds$$

$$u_{xx} = \int_0^t v_{xx}(x,t;s)\,ds$$

Then $u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t \underbrace{\left(v_{tt}(x,t;s) - c^2 v_{xx}(x,t;s)\right)}_{0} ds$ ■

⌞

There is an exercise 3 to solve inhomogeneous wave equation in a different manner (using Green's theo

Thus, the solution to (**) looks like:

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z,t)\,dz + \frac{1}{2c}\int_0^t \int_{x-c(t-T)}^{x+c(t-T)} h(z,T)\,dz\,dT.$$

Remark: Duhamel principle is a powerful (universal) method of solving inhomogeneous problems. It works for ODEs, heat equation etc....

Mixed initial-boundary value problem

Consider a string of a guitar

$$u_{tt} - c^2 u_{xx} = h(x,t)\ , \quad x \in [a,b]$$

$$\left.\begin{array}{l} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{array}\right\} \text{"initial" conditions} \qquad (***)$$

$$\left.\begin{array}{l} u(a,t) = a(t) \\ u(b,t) = b(t) \end{array}\right\} \text{"boundary" conditions}$$

One can solve this problem explicitly using Fourier sums (we will do it later). But let us show tha even if we do not know the exact form of solution, we can prove $\exists$ and $!$

Thm (uniqueness for wave equation) There exists at most one function $u \in C^2([a,b]\times[0,T$ solving $(***)$.

Proof:

▶ We will prove using "energy method".

Suppose w, v are two solutions of $(***)$

Then $u = w - v$ is a solution to homogeneous problem:
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = u_t(x,0) = 0 \\ u(a,t) = u(b,t) = 0 \,. \end{cases}$$

Let us show that $u \equiv 0$.

Define the "energy":
$$I(t) = \frac{1}{2} \int_a^b \Big(\underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{u_t^2} + \underset{\substack{\uparrow \\ \text{potential} \\ \text{energy}}}{c^2 u_x^2} \Big)\, dx$$

How does $I(t)$ change with time?

$$\frac{dI(t)}{dt} = \frac{1}{2} \int_a^b \Big(2 u_t \cdot \underbrace{u_{tt}}_{= c^2 u_{xx}} + c^2 \cdot 2 \cdot u_x \cdot u_{xt} \Big)\, dx =$$
$$= c^2 \int_a^b \big(u_t \cdot u_{xx} + u_x \cdot u_{xt} \big)\, dx = c^2 \int_a^b \frac{d}{dx} (u_t \cdot u_x)\, dx$$
$$= c^2 u_t \cdot u_x \Big|_a^b = 0 \quad \Rightarrow \quad I(t) \equiv \text{const}$$

$I(0) = 0 \Rightarrow I(t) \equiv 0$. Thus $u_t \equiv 0$, $u_x \equiv 0 \Rightarrow u \equiv$ con

As $u(x,0) = 0 \Rightarrow u \equiv 0$ ∎

__Thm__ (existence of solution to a wave equation)
There exists a solution to problem $(***)$ $u \in C^3([a,b] \times [0,T])$

__Proof__:

For simplicity, let $c=1$ (the same thing for $c \neq 1$, let it be an exercise)

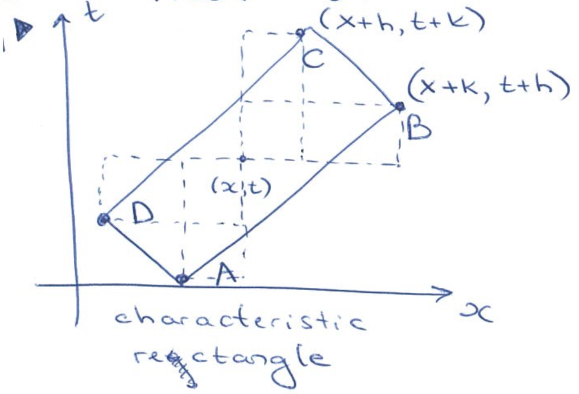
Before we prove, let me formulate and prove

__Useful__ __lemma__: $u(x,t) \in C^3$. The following statements are equivalent:

(1) u satisfies PDE $u_{tt} - u_{xx} = 0$

(2) u satisfies the difference equation
$$u(x-k, t-h) + u(x+k, t+h) = u(x-h, t-k) + u(x+h, t+k)$$
$\forall (x,t) \in \mathbb{R} \times \mathbb{R}$ and $k, h > 0$. See remark below^1.

Proof of lemma:



characteristic rectangle

$\boxed{(1) \Rightarrow (2)}$ $u(x,t) = f(x-t) + g(x+t)$

Notice that f and g separately satisfy the difference eq: $f(x-k-t+h) + f(x+k-t-h)$
$= f(x-h-t+k) + f(x+h-t-k)$
$\Rightarrow u$ satisfies the diff. eq.

$\boxed{(2) \Rightarrow (1)}$ Let $h=0$.

$u(x,t)$ satisfies the difference equation in (2).
Subtrack $2u(x,t)$ and devide for k^2:

$$\frac{u(x-k,t) - 2u(x,t) + u(x+k,t)}{k^2} = \frac{u(x,t-k) - 2u(x,t) + u(x,t+k)}{k^2}$$

By Taylor expansion, we get

$$u(x-k,t) = u(x,t) - k u_x(x,t) + \frac{1}{2} k^2 u_{xx}(x,t) + O(k^3)$$ here we use that u is from C^3
$$u(x+k,t) = u(x,t) + k u_x(x,t) + \frac{1}{2} k^2 u_{xx}(x,t) + O(k^3)$$

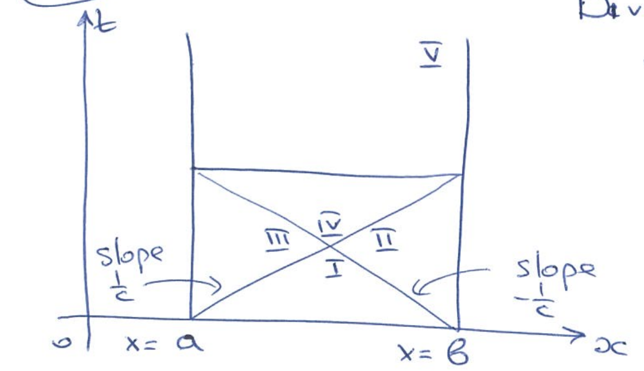
So we have $u_{xx} + O(k) = u_{tt} + O(k)$, $k \to 0$

As a limit we get the wave equation. ∎

Intuitively formula (2) is very clear. Indeed the left hand side is a discrete analog of u_{xx} (if $h=0$): $u(x-k,t) + u(x+k,t) =$
$= (u(x-k,t) - u(x,t)) - (u(x,t) - u(x+k,t)) \sim -u_x(x-k) + u_x(x) \sim u_{xx}(x)$

Proof of existence: simple geometric idea.

$c=1$

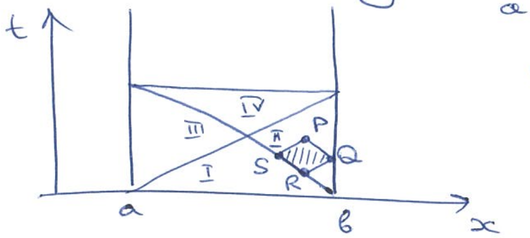


Divide the domain $\Omega = [a,b] \times \mathbb{R}_+$ into 5 pieces as shown on the picture draw a line with slope $\frac{1}{c}$ from point a, and a line with slope $-\frac{1}{c}$ from point b; and consider a rectange such that these two lines are its diagonals.

Then the following observations are valid.

I. The solution in region I is completely determined by D'Alembert formula.

II. To construct solution at any point P in region II we use the following characteristic rectangle (see picture) and use useful lemma.



$$u(P) + u(R) = u(S) + u(Q)$$

boundar cond.

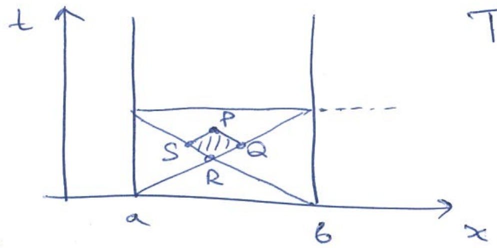
$$\Rightarrow u(P) = u(S) + u(Q) - u(R)$$

we already know!

Thus we know u in region II.

III. Analogously, we construct u in region III.

IV. To construct u in region IV we use the following characteristic rectangle and use useful lemma.



Thus, we have constructed the solution for $x \in [a,b]$, $t \in [0, \frac{b-a}{c}]$

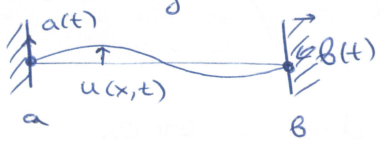
Repeat this procedure to construct u for all $t>0$ ■

Exact solution to : $$\begin{cases} u_{tt} - c^{2} u_{xx} = 0 \\ u|_{x=0} = u|_{x=\pi} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Lecture 4

Last time: mixed initial-boundary value probler

guitar string oscillation



$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \text{(ic)} \qquad \begin{cases} u(a,t) = a(t) \\ u(\beta,t) = \beta(t) \end{cases} \text{(bc)}$$

- We proved $\exists!$ of solution
- $c = \sqrt{T/g}$, g-densit, T-tensic

Today let us find explicitly the solution to:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ u(a,t) = u(\beta,t) = 0. \end{cases}$$

We will do it using Fourier series.

Small reminder on Fourier series

Def: Fourier series of function f is a representation:

$$(1) \quad \boxed{f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}}, \quad c_n \in \mathbb{C}, \quad x \in \mathbb{R}, \quad f: \mathbb{R} \to \mathbb{R}$$

(in fact, f is periodi with period(2

Here the series converges absolutely.
It is enough to assume that $\sum_{n\in\mathbb{Z}} |c_n| < +\infty$

Rmk: if f is real $\Rightarrow$ $f(x) = \overline{f}(x) = \sum_{n\in\mathbb{Z}} \overline{c}_n e^{-inx} =$

$$= \sum_{n\in\mathbb{Z}} \overline{c}_{-n} e^{inx}$$

$$\Rightarrow \quad \boxed{\overline{c}_{-n} = c_n}$$

$\Rightarrow$ let's define $c_0 = \frac{a_0}{2}$, $c_n = (a_n - i\beta_n)\frac{1}{2}$

$$c_{-n} = (a_n + i\beta_n)\frac{1}{2}$$

Thus,
$$f(x) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(a_n e^{inx} + c_{-n} e^{-inx} \right) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left(\frac{a_n}{2} \cos(n \right.$$
$$\left. + \frac{\beta_n}{2} \sin(nx) \right) + \sum_{n\in\mathbb{N}} \left(\frac{a_n}{2} \cos(nx) + \frac{\beta_n}{2} \sin(nx) \right)$$

$$\Rightarrow \quad \boxed{f(x) = \frac{a_0}{2} + \sum_{n\in\mathbb{N}} \left[a_n \cos(nx) + \beta_n \sin(nx) \right]} \qquad (2)$$

Thm 1: For a function given by Fourier series (1), we can define a coefficient:

Fourier coef. $\boxed{c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx}$ (3)

Proof:

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}} c_m e^{imx} e^{-inx}\,dx = $$

/ can change $\int$ and as the sum is abs convergent /

$$= \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = 2\pi c_n$$

because $$\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \begin{cases} 2\pi, & m=n \\ 0, & m\neq n \end{cases} \blacksquare$$

Observation:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\,; \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx\,; \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$$

Remark: from point of view of functional analysis: consider a Hilbert space

$$L^2[0,2\pi] = \underbrace{\{ f : [0,2\pi] \to \mathbb{C} \text{ - measurable} : \int_0^{2\pi} |f(x)|^2 dx < +\infty \}}\,/\sim$$

where $f\sim g$ means $f=g$ w.r.t. Lebegue measure

that is $\mu\{x : f(x)\neq g(x)\} = 0$.
$\uparrow$ Lebegue measure on $\mathbb{R}$.

Then
- $\left(\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2} = \|f\|_{L^2[0,2\pi]}$ - norm
- $\int_0^{2\pi} f(x)\overline{g(x)}\,dx = \langle f,g\rangle_{L^2[0,2\pi]}$ - scalar product

Actually, $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal basis
($\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormall basis)

And $\forall f \in L^2[0,2\pi]$ can be represented by formula (1)

$$\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right\rangle = \delta_{nm} \leftarrow \text{Kronecker symbol}$$

$$\delta_{nm} = \begin{cases} 1, & n=m \\ 0, & n\neq m \end{cases}$$

in the sense of L_2: $\int_0^{2\pi} |f(x) - \sum_{|n|\leq N} c_n e^{inx}|\,dx \xrightarrow[N\to\infty]{} 0$

In finite dimensions we have $u \in \mathbb{R}^d$ and $\{e_1, .., e_d\}$ basis, then $\exists!\ u_k : \quad u = \sum_{k=1}^{d} u_k e_k$

To find u_k we just take scalar product with e_n

$$\langle u, e_n \rangle = \sum_{k=1}^{d} u_k \cdot \langle e_k, e_n \rangle = u_n \langle e_n, e_n \rangle$$

$$\Rightarrow \quad u_n = \frac{\langle u, e_n \rangle}{\langle e_n, e_n \rangle} .$$

For infinitedimensional space it is similar.

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx} \quad \Big| \cdot \langle \cdot , e^{imx} \rangle$$

$$\langle f(x), e^{imx} \rangle = c_m \underbrace{\langle e^{imx}, e^{imx} \rangle}_{\overset{\shortparallel}{2\pi}} .$$

$$\Rightarrow \quad c_m = \frac{1}{2\pi} \langle f(x), e^{imx} \rangle .$$

- The same story for $\{1, \cos(nx), \sin(nx)\}$ - basis in $L^2[0, 2\pi$ for real-valued f.

Thm 2: Let $f(x) \in C^\infty(S^1)$ - smooth periodic function on a circle $S^1 = [0, 2\pi] / \{0 = 2\pi\}$

Then for any $a \geq 0$ there exists a constant C (which depend on f and a, but independent of n) such that

$$|c_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \right| \leq C \cdot |n|^{-a} \quad \text{for } |n| \neq 0$$

(c_n goes to 0 very fast - faster than any polynomial

Proof:

$a = 0$: $\quad |c_n| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) e^{-inx}| dx = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx}_{=: C}$ proved!

$a = 1$: integrate by parts:

$$|c_n| = \left| \cancelto{0}{\frac{1}{2\pi} f(x) \frac{e^{-inx}}{-in} \Big|_{-\pi}^{\pi}} + \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \right| \leq$$

$$\leq \frac{1}{n} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)| dx =: \frac{C}{n} .$$ proved!

And so on....

$\blacksquare$

Corollary: For any $f(x) \in C^{\infty}(S^1)$ the corresponding Fourier series $\sum\limits_{n \in \mathbb{Z}} c_n e^{+inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$, converges for all $x \in \mathbb{R}$.

Proof:

Γ▶ Absolute convergence is clear:

$$\left|\sum c_n e^{inx}\right| \leq \sum |c_n| \underset{1}{\underset{\shortparallel}{\left|e^{inx}\right|}} \leq \sum \frac{c}{n^2} < +\infty \quad \blacksquare$$

L

Summing up:

Thm 1: $f(x) = \sum c_n e^{inx} \Rightarrow c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,d$

Thm 2 + Corrollary: $f \in C^{\infty}(S^1) \Rightarrow$ we can write a serie $\sum c_n e^{-inx}$ and it converges.

Does this series always converge to $f(x)$?

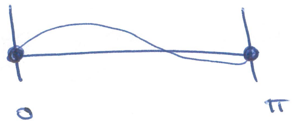
Not always! (in general)

Thm 3 (without proof):

$f \in C^2(S^1)$, then $f(x) = \sum\limits_{n \in \mathbb{Z}} c_n e^{inx}$, $c_n = \frac{1}{2\pi}\int_{-\pi} f(x$ e^{-i} d

e.g. Arnold's book

Solution to



$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_{x=0} = u|_{x=\pi} = 0 \\ u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

$\varphi(0) = \varphi(\pi) = 0$

$\psi(0) = \psi(\pi) = 0$

Spectral method: (we will need to solve auxiliary eigenvalue problem)

we have constant coefficients, let's try solution of the form:

$$u(x,t) = \varphi(x)\cdot e^{\lambda t} \quad \text{for some } \lambda \in \mathbb{C}$$

$$\lambda^2 \cdot \varphi(x) \cdot \cancel{e^{\lambda t}} - c^2 \cdot \varphi''(x) \cdot \cancel{e^{\lambda t}} = 0$$

So we have $\begin{cases} \varphi'' + \mu\varphi = 0 \\ \varphi(0) = \varphi(\pi) = 0 \end{cases}$ for $\mu = -\frac{\lambda^2}{c^2} \in \mathbb{C}$

→ eigenvalue problem

$$\begin{cases} \varphi'' + \mu\varphi = 0 \\ \varphi(0) = \varphi(\pi) = 0 \end{cases}$$ Let's find all μ for which the solution exists.

Case $\mu < 0$: $\varphi'' + \mu\varphi = 0 \Rightarrow \varphi(x) = A\, e^{\sqrt{|\mu|}x} + B\, e^{-\sqrt{|\mu|}x}$

$$\begin{matrix} \varphi(0) = 0 \\ \varphi(\pi) = 0 \end{matrix} \Rightarrow \begin{cases} A + B = 0 \\ A \cdot e^{\sqrt{\mu}\pi} + B\, e^{-\sqrt{\mu}\pi} = 0 \end{cases} \Rightarrow A = B = 0$$

only trivial solution $\varphi \equiv 0$.

Case $\mu = 0$: $\varphi'' = 0 \Rightarrow \varphi(x) = Ax + B$

$\varphi(0) = \varphi(\pi) = 0 \Rightarrow A = B = 0 \Rightarrow \varphi \equiv 0$.

Case $\mu > 0$: $\varphi'' + \mu\varphi = 0 \Rightarrow \varphi(x) = A \cdot e^{i\sqrt{\mu}x} + B \cdot e^{-i\sqrt{\mu}x}$

better $\varphi(x) = A\sin(\sqrt{\mu}x) + B\cos(\sqrt{\mu}$

$\varphi(0) = 0 \Rightarrow B = 0 \Rightarrow \varphi(x) = A\sin(\sqrt{\mu}x)$

$\varphi(\pi) = 0 \Rightarrow A\sin(\sqrt{\mu}\,\pi) = 0 \Rightarrow \sqrt{\mu}\,\pi = \pi k,\ k \in \mathbb{Z}$

$\mu = k^2,\ k \in \mathbb{Z}$

Obs: μ can be only real and positive

$\varphi'' + \mu\varphi^* = 0 \ \Big|\ \cdot \langle \cdot, \varphi \rangle_{L^2}$

$\int_0^{2\pi} \varphi'' \cdot \varphi + \mu\varphi^2 = 0$ $\qquad$ $\mu \cdot \int_0^{2\pi} \varphi^2 = \int_0^{2\pi} (\varphi')^2$

$$\mu = \frac{\int_0^{2\pi} (\varphi')^2}{\int_0^{2\pi} \varphi^2} > 0.$$

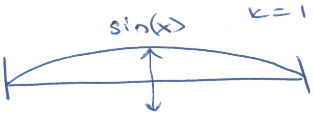
Then $\lambda^2 = -c^2k^2$

$\lambda = ick$ and we have infinitely many

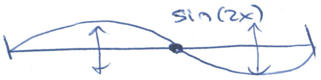
solutions: $u_k(x,t) = \sin(kx) \cdot \underbrace{e^{ikt}}_{\text{complex}}$

we are interested only in real solutions, so we can consider any sums like this:

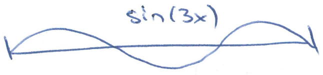
$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx)\left[A_k \cos(ckt) + B_k \sin(ckt)\right]$$

=1


main mode
(fundamental tone)

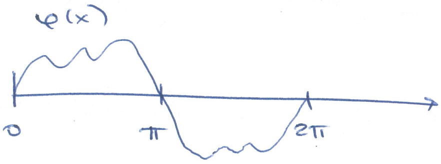
k=2


overtone

k=3


Let us show that this solution is general.

First, notice that $\varphi(x)$ can be represented only as a sum of $\sin(kx)$ in its Fourier series.



→ continue $\varphi(x)$ to interval $[\pi, 2\pi]$ oddly $\Rightarrow$
$$\varphi(x) = \sum_{k=1}^{\infty} A_k \sin(kx)$$

→ similar with $\psi(x)$:
$$\psi(x) = \sum_{k=1}^{\infty} B_k \sin(kx)$$

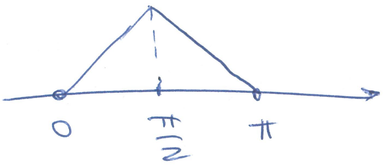
Second, we have
$$u(x,t) = \sum_{k=1}^{\infty} \sin(kx) \left(A_k \cos(kt) + B_k \sin(kt) \right)$$

$$u(x,0) = \varphi(x) = \sum A_k \sin(kx) \Rightarrow A_k = a_k$$
$$u_t(x,0) = \psi(x) = \sum ck B_k \sin(kx) \Rightarrow ck B_k = b_k \Rightarrow B_k = \frac{b_k}{kc}.$$

$$\Rightarrow \quad u(x,t) = \sum_{k=1}^{\infty} \sin(kx) \left(a_k \cos(kt) + \frac{b_k}{kc} \sin(kt) \right).$$

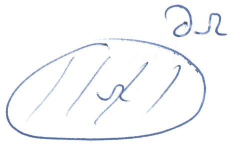
where $a_k = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin(kx)\, dx$, $b_k = \frac{2}{\pi} \int_0^{\pi} \psi(x) \sin(kx)\, d$

Exercise: Find a Fourier series solution to
$$\varphi(x) = \begin{cases} x, & x \in [0, \frac{\pi}{2}] \\ \pi - x, & x \in [\frac{\pi}{2}, \pi] \end{cases}$$



$$\psi(x) \equiv 0.$$

Various space dimensions: $\Omega \subset \mathbb{R}^d$



$$u_{tt} - c^2 \Delta u = 0$$
$$u|_{\partial\Omega} = 0$$

One can look for solutions of the form: $u(r,t) = \varphi(r) \cdot e^{i\omega t}$ and have the following eigenvalue problem:
$$\begin{cases} \Delta\varphi + \frac{\omega^2}{c^2} \varphi = 0 \\ \varphi|_{\partial\Omega} = 0 \end{cases}$$

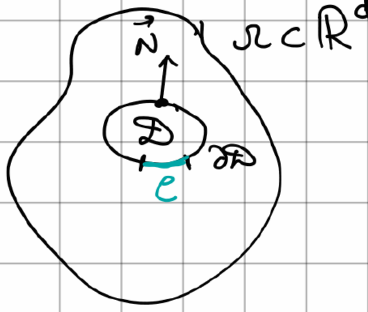
For compact Ω with smooth boundary, we usually have a family $\{\frac{\omega_k^2}{c^2}, \varphi_k\}_{k \in \mathbb{N}}$ of eigenvalues/functio

$$u(x,t) = \sum \varphi_k(x) \left(A_k \sin(\omega_k c t) + B_k \cos(c\omega_k t) \right).$$

$\boxed{\text{Lecture 5:}}$ Conservation & Balance laws

Plan:
1. General definition
2. Example 1: fluid dyn (conservation of mass)
3. Example 2: scalar conservation law

① Balance law



$\mathcal{D} \subset \Omega$ with Lipshitz boundary (smooth)

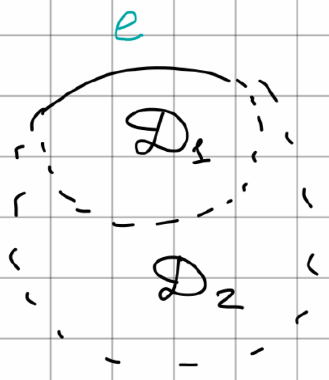
$\vec{N}$ - normal vector towards the exterior of the domain $\mathcal{D}$

$$\boxed{\text{production in } \mathcal{D} = \text{flux through the boundary of } \partial\mathcal{D}}$$

- production in $\mathcal{D}$ is some measure (Radon) $\mathcal{P}$
- flux $$F_{\mathcal{D}}(e) = \int_{e} q_{\mathcal{D}}(x)\, dS(x)$$

$$\mathcal{P}(\mathcal{D}) = \int_{\partial\mathcal{D}} \underline{q_{\mathcal{D}}(x)}\, dS(x) \qquad (*)$$

Assume:



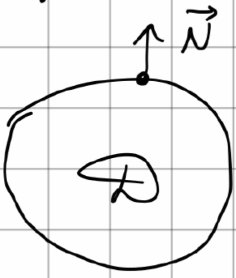
$q_{\mathcal{D}_1}(x) = q_{\mathcal{D}_2}(x)$

$\forall x \in e$

Take-home (Tuesday) $$\boxed{\operatorname{div} A = 0}$$

Miracles :

Consequences of (*):



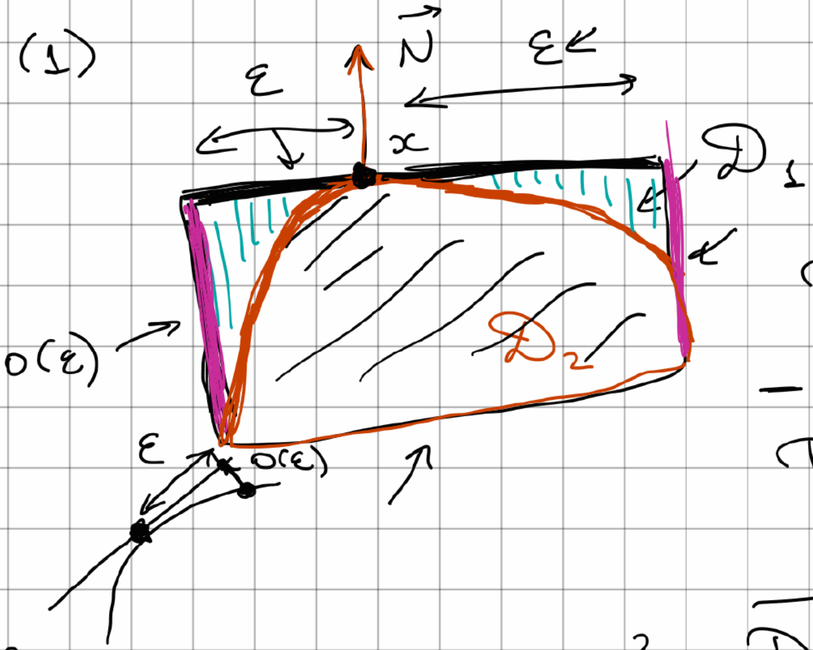
(1) $\exists \; a_{\vec{N}}(x) = q_{\mathcal{D}}(x)$ $\forall x \in \Omega$ for any $\mathcal{D} \subset \Omega$ s.t. $\mathcal{D}$ has $\vec{N}$ as a normal verctor at x.

(2) $\exists \; \vec{A}(x) : \Omega \to \mathbb{R}^d$: $a_{\vec{N}}(x) = \vec{A}(x) \cdot \vec{N}$

(3) $\exists$PDE: $\operatorname{div} \vec{A} = P$

$$P(\mathcal{D}) = \int_{\partial \mathcal{D}} \underbrace{q_{\mathcal{D}}(x)}_{\substack{\parallel \\ \vec{A}(x)\cdot\vec{N}}} dS(x)$$

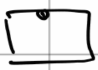
(1)



$\varepsilon \to 0$

$q_{\mathcal{D}_1}(x) \overset{?}{=} q_{\mathcal{D}_2}(x)$

(*)

$P(\mathcal{D}_1) = \int q_{\mathcal{D}_1}(x) dS$


$-$

$P(\mathcal{D}_2) = \int q_{\mathcal{D}_2}(x) dS$


$\varepsilon^2 \sim P($$) = \int q_{\mathcal{D}_1}(x) dS$

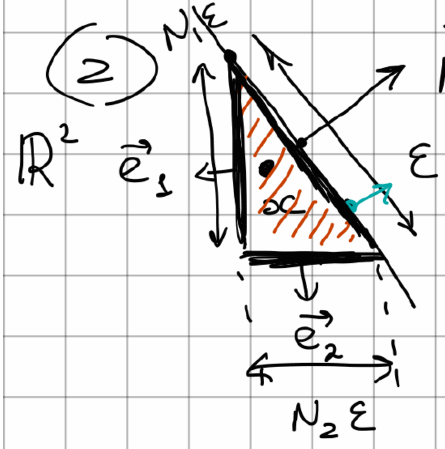

$- \int q_{\mathcal{D}_2}(x) dS$


$\int \sim o(\varepsilon)$

$\Rightarrow \int q_{\mathcal{D}_1}(x) dS(x) = \int q_{\mathcal{D}_2}(x) dS(x)$


$$\Rightarrow \quad \exists \; a_{\vec N}(x) = q_{\mathcal{D}}(x) \qquad \underline{\text{Cauchy}}$$

②



$$\vec N = (N_1, N_2)$$

$$\varepsilon^2 \sim \mathcal{P}(\triangle) = \int_{\Delta} a_{\vec N}(x)\,dx$$

$$a_{\vec N}(x)\cdot \underline{\underline{\cancel{\varepsilon}}} = a_{e_2}(x)\cdot \underline{\underline{N_2\cancel{\varepsilon}}} + a_{e_1}(x)\cdot N_1\cancel{\varepsilon}$$

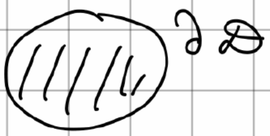
$$\Rightarrow \quad a_{\vec N}(x) = a_{e_1}(x)\,N_1 + a_{e_2}(x)\,N_2$$

$$\Rightarrow \quad a_N(x) = \vec A(x)\cdot \underline{\underline{\vec N}}$$

$$A(x) = (a_{e_1}, a_{e_2})$$

③ $$\int_{\partial\mathcal{D}} \vec A(x)\cdot\vec N \, dS(x) = \int_{\mathcal{D}} \operatorname{div}(\vec A)\,dx$$

↑ Green-Gauss theorem



$\|$

$$\mathcal{P} = \int_{\mathcal{D}} p(x)\,dx$$

$$\Rightarrow \quad \operatorname{div}(\vec A) = p \quad \text{– balance law}$$

$$\boxed{\operatorname{div}(\vec A) = 0} \quad \text{– conservation law}$$

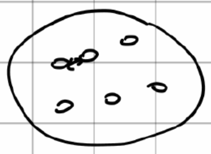
Dafermos

Example 1 : Fluid flow, continuum mechanics

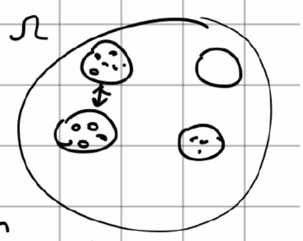
different scales

1. atoms/molecules

2. representative small volume



3. domain (macroscale)



- Eulerian vs. Lagrangian point of view

Eulerian : (x,t) - fix

- velocity : $u(x,t) = (u_1, \dots, u_d) : \mathbb{R}^d \times [0,\infty)$ $\mathbb{R}^{d}$
  has units $\left[\frac{L}{T}\right]$
- density : $\rho(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[\frac{M}{L^d}\right]$
- pressure : $p(x,t) : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$
  with units $\left[M L^{-d+2} T^{-2}\right]$

Lagrangian : particles, $a \in \mathbb{R}^d$
trajectories of particles

flow map $X(t,a) = (X_1, \dots, X_d)$ - position of particle a at time t

$$(**) \begin{cases} \partial_t X(t,a) = u(t, X(t,a)) & \leftarrow \text{ODE} \\ X(0,a) = a \end{cases}$$

ODE theory (Cauchy-Lipschitz theorem) :
$u \in C_t \mathrm{Lip}_x \Rightarrow \exists!$ solution to $(**)$

$X(t,\cdot)$ – is C^1-diffeo : $\mathbb{R}^d \to \mathbb{R}^d$

Define inverse: $A(t,x) : [0,+\infty)\times\mathbb{R}^d \to \mathbb{R}^d$

$$A(t, X(t,a)) = a \qquad X(t, A(t,x)) = x$$

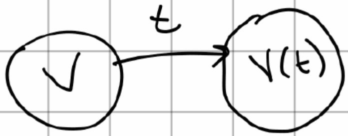
$$\forall x, a \in \mathbb{R}^d$$

"back-to-labels" map (a – "labels")

Incompressibity condition : "div $u = 0$"

Take $V \subset \Omega$ – volume of fluid

$$V(t) = X(t,V) = \{X(t,a) : a \in V\}$$



Def: velocity field is called incompressible if

$\longrightarrow$ $|V(t)| \equiv |V|$

$\uparrow$ Lebesgue measure of V

Lemma: $u \in C_t \mathrm{Lip}_x$

u is incompressible $\Longleftrightarrow$ div $u = 0$ (u is divergence-free)

Proof:

$\triangleright$ $V(t) = \int_{V(t)} 1 \cdot dx$; $a \in V \subset \mathbb{R}^d$

$$\int_{V(t)} f(x,t)\,dx = \int_V f(\underset{a \mapsto X(t,a)}{X(t,a)}, t) \cdot \underbrace{\boxed{\det(\nabla_a X)}^{(t,a)}}_{= J(t,a)} da$$

$$\longrightarrow J(t,a) = \sum_{i_1,\dots,i_d=1}^{d} \underline{\varepsilon_{i_1\dots i_d}} \frac{\partial X_{i_1}}{\partial a_1} \cdot \ldots \cdot \frac{\partial X_{i_d}}{\partial a_d}$$

Exercise: $\partial_t J(t,a) = J(t,a)\cdot(\operatorname{div} a)(t, X(t,a))$

Corollary: $J(t,a) \equiv 1 \iff \operatorname{div}(u) = 0$

$$\hookrightarrow J(t,a) = J(0,a) \cdot e^{\int_0^t (\operatorname{div} a)(s, X(s,a))\,ds}$$

$$J(0,a) = 1$$

$$J(t,a) = J(0,a) \quad \forall t \Rightarrow \operatorname{div}(a) = 0$$

$$V(t) = \int_{V(t)} 1\,dx = \int_V J(t,a)\,da \underset{\substack{\uparrow \\ \text{iff } \operatorname{div}(a)=0}}{=} \int_V da = V$$

L ■

Transport equation

Let's $f(t,x) : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ - scalar

Eulerian: $\partial_t f$ - change of f at (t,x)

Lagrangian: $\partial_t f(t, X(t,a)) =: D_t f$ - convective derivative

$$\overset{\shortparallel}{\partial_t f + u \cdot \nabla f}$$

Thm (transport thm): $\textcircled{V} \xrightarrow{t} \textcircled{V(t)}$

u - velocity field, $u \in C^1$; f be C^1

$V(t)$ is pushforward of V by the flow map $X(t,a)$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_{V(t)} \left(\partial_t f + \operatorname{div}(fu)\right)(t,x)\,dx$$

Proof:

Γ▶ $$\int_{V(t)} f(x,t)\,dx = \int_V \overset{\downarrow}{f(X(t,a),t)}\ \underline{\overset{\downarrow}{J(t,a)}}\,da$$

$$\frac{d}{dt}\left(\int_{V(t)} f(x,t)\,dx\right) = \int_V \underline{(D_t f)}\,(X(t,a),t)\; J(t,a$$

$$+ \int_V f(X(t,a),t)\cdot \underbrace{\partial_t J(t,a)}_{J(t,a)\cdot \mathrm{div}(u)}\,da =$$

$$= \int_V \Big(\partial_t f + \underbrace{u\cdot\nabla f + f\cdot \mathrm{div}(u)}_{\mathrm{div}(uf)}\Big)\,(X(t,a),t)\cdot J(t,a\; da$$

$$= \int_V \big(\partial_t f + \mathrm{div}(fu)\big)\,(X(t,a),t)\cdot J(t,a)\,da =$$

$$= \int_{V(t)} \big(\partial_t f + \mathrm{div}(fu)\big)\,dx \qquad \blacksquare$$

Conservation of mass : $\rho(x,t)$

$$m(t,V) = \int_V \rho(x,t)\,dx$$

$$\frac{d}{dt}\, m(t,V(t)) = 0$$

Thm : conservation of mass is equivalent to the following integral eq:

$$\int_{V(t)} \big(\rho_t + \mathrm{div}(\rho u)\big)\,dx = 0$$

If ρ_t and $\mathrm{div}(\rho u)$ are C, then

$$\boxed{\rho_t + \mathrm{div}(\rho u) = 0}$$

Rmk : → scalar transport eq

Proof:

$$\ulcorner\; 0 = \frac{d}{dt} m(t, X(t,a)) = \frac{d}{dt}\int_{V(t)} \rho(x,t)\,dx$$

$$= \int_{V(t)} \underbrace{\left(\rho_t + \operatorname{div}(\rho u)\right)}_{\text{are continuous}} dx .$$

$$f(x) = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\,dy$$

$$\llcorner \Rightarrow \quad \rho_t + \operatorname{div}(\rho u) = 0. \qquad \blacksquare$$

Rmk: $0 = \rho_t + \operatorname{div}(\rho u) = \rho_t + u\cdot\nabla\rho + \operatorname{div}(u)\rho$

incompressibility $\Rightarrow$ $\operatorname{div}(u) = 0$

$$\rotatebox{90}{=}$$

$$D_t \rho = \rho_t + u\cdot\nabla\rho = 0$$

$$\frac{d}{dt}\left(\rho(t, X(t,a))\right) = 0$$

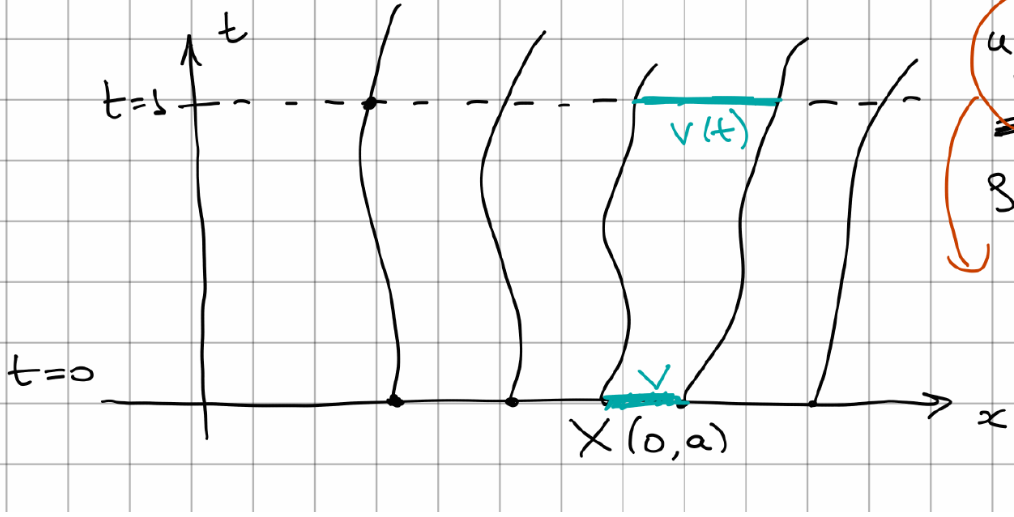
$$\Rightarrow \quad \rho(t, X(t,a)) = \text{const.} \hookleftarrow$$

Next time

$\rho_t + (\underline{\rho u})_x = 0$

$u = u(\rho)$

$u(\rho) = \frac{\rho}{2}$ $\Rightarrow$ Burgers



$u \in C_t \underline{\mathrm{Lip}_x}$

linear in ρ

$\rho_t + \underline{u}\cdot\nabla\rho = 0$

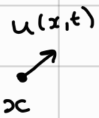
$\boxed{\text{Lecture 6}}$

Last time: Balance laws: $\text{div} A = P$

Conservation laws: $\text{div} A = 0$

Example 1: fluid flow: $a \to X(t,a)$ – flow map under velocity field

$a \in \mathbb{R}^d$



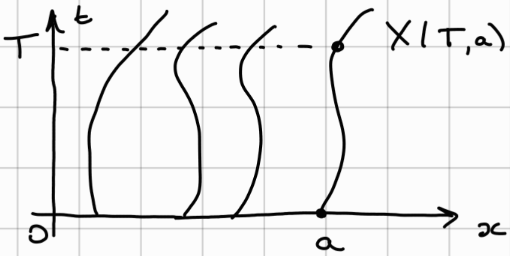
$u(t,x)$ – velocity field

$\rho(t,x)$ – density

$$\begin{cases} \partial_t X = u(X,t) \\ X(0,a) = a \end{cases}$$

- Conservation of mass = scalar transport equation

$$\boxed{\partial_t \rho + \text{div}(\rho u) = 0}$$

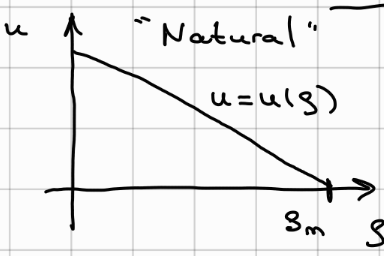


trajectories do not intersect for given $u \in C^0_t \, \text{Lip}_x$

Rmk: $\begin{cases} \text{div } u = 0 \\ \partial_t \rho + \text{div}(\rho u) = 0 \end{cases}$

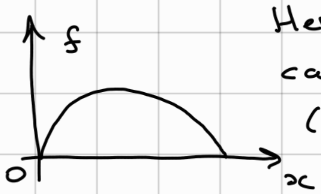
$\Rightarrow$ $\rho(t, X(t,a)) = \text{const}$ density is conserved along the trajectory for incompressible flow

Example 2: traffic flow: cars choose their velocity depending on "density" of cars nearby

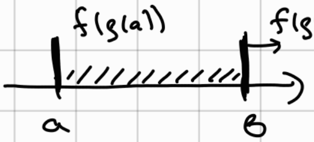


$\Rightarrow$ $\boxed{\rho_t + f(\rho)_x = 0}$ scalar conservation law

Here $f(\rho)$ is called flux (flow function)



ρ_m – density of cars corresp. to "bumper-to-bumber"



$$\frac{d}{dt} \int_a^b \rho(x,t)\,dx = f(\rho(b)) - f(\rho(a))$$

Rmk: 1) taking $u(\rho) = \frac{\rho}{2}$ $\Rightarrow$ Burgers eq: $\rho_t + \left(\frac{\rho^2}{2}\right)_x = 0$

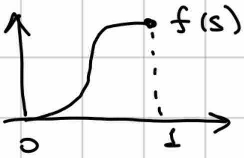
We will analyze it in detail today.

2) for oil recovery the simplest 1-dim model for displacement water-oil is again

$$S_t + (f(S))_x = 0 \quad \text{for}$$

S - water saturation

f(S) - fractional flow function



$f(0)=0$
$f(1)=1$
f ↑ and S-shaped

- One can easily create more sophisticated models such as: take drivers anticipation into account

If a driver observe an upstream increase in the density, they show a tendency to brake slightly

$$u - v(\rho) \sim -\rho_x$$

The simplest law: $u = v(\rho) - \varepsilon \rho_x$, $0 < \varepsilon \ll 1$

which leads to the "weakly" parabolic eq:

$$\rho_t + f(\rho)_x = \varepsilon(\rho\rho_x)_x$$

Example 3: wave equation ! $u_{tt} - c^2 u_{xx} = 0$

$\mathrm{div}(u_t, -c^2 u_x) = 0$

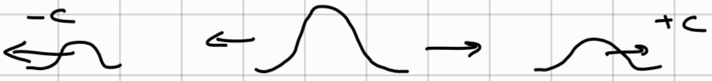
Consider $U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + A U_x = 0$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Indeed, this is just: $\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - c^2 u_{xx} = 0 \end{cases}$

Eigenvalues of A: $\det\begin{vmatrix} 0-\lambda & -1 \\ -c^2 & -\lambda \end{vmatrix} = \lambda^2 - c^2$, $\lambda_\pm = \pm c$

They correspond to propagation modes:



This is general fact that we will see in the future:

$U \in \mathbb{R}^d$, $F: \mathbb{R}^d \to \mathbb{R}^d$ $\quad U_t + (F(U))_x = 0$ - system of conservation laws

Then for "smooth" solutions we have:

$$U_t + \underbrace{F'(U)}_{\text{eigenvalues of this matrix play an important role !}} \cdot U_x = 0$$

If they are real, they correspond to velocity of propagation of waves.

Example 4 : isentropic (= constant entropy) gas dynamics
(p-system)

in Lagrangian coordinates : $\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = 0 \end{cases} \Rightarrow v_{tt} + p(v)_{xx} = 0$

Rmk : $v_t = u_x \Rightarrow$ (in a simply connected regions)
$\exists\ \varphi :\quad v = \varphi_x,\ u = \varphi_t$

$$\Rightarrow \quad \varphi_{tt} + (p(\varphi_x))_x = 0$$

$$\varphi_{tt} + p'(\varphi_x)\cdot \varphi_{xx} = 0$$ – nonlinear wave equation

And many other examples :

- conservation of mass
  conservation of momentum $\Big\} \Rightarrow \begin{cases} \partial_t u + u\cdot\nabla u = \nabla p + f \\ \operatorname{div}(u) = 0 \end{cases}$

  This is Euler equations for ideal fluid (1755, second PDE ?)
- Navier-Stokes eqs (1845): adds viscosity
  $$\partial_t u + (u\cdot\nabla)u - \nu\Delta u = \nabla p$$
- gas dynamics
- electromagnetism (Maxwell eqs)
- magneto-hydrodynamics (M.H.D.) – motion of fluid in the presence of electromagnetic field (think of a Sun)

Etc ....

Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \qquad\qquad u_t + u\cdot u_x = 0$$

Observation 1: if $u \in C^1$ for all $t>0$, then u is monotonically nondecreasing in x for all $t>0$.

$$u\,(X(s),\, t(s)) = \text{const}$$

$$u_t \cdot t_s + u_x \cdot X_s = 0$$

$$\begin{cases} t_s = 1 \\ X_s = u \end{cases} \Rightarrow$$ u = const on straigh lines $x = x_0 + u_0(x_0)\,t$

If $u \in C^1$ for $\forall\, t>0$, then characteristics should not intersect $\Rightarrow$ $u_0(x_1) < u_0(x_2)$ if $x_1 < x_2$ $\Rightarrow$ u_0 is non-decreasing ($u(x,0)$) $\Rightarrow$ $u(x,t)$ is non-decreasing in x

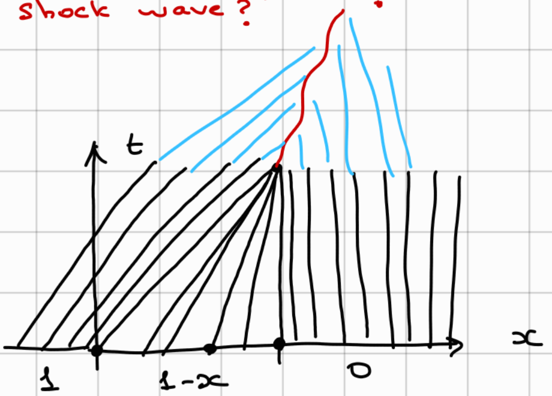
Exercise 2 from list 1:

$$x = x_0 + (1-x_0)\,t$$

$t=1$: $x = 1$

At $t=1$ there is a blow-up

exists a unique shock wave?



Rmk: In general scalar conservation law:

$$u_t + f(u)_x = 0$$

$$u_t + \underline{\underline{f'(u)}} \cdot u_x = 0$$

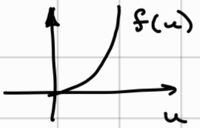
Characteristics are $x = x_0 + f'(u_0(x_0))\,t$

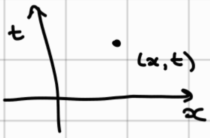
$u \in C^1$ $\forall\, t>0 \Rightarrow f'(u_0(x_1)) < f'(u_0(x_2))$ if $x_1 < x_2$, otherwise characteristics will intersect that leads to a blow-up!

So no matter how smooth f and u_0 are, the solution $u(x,t)$ must become discontinuous

This is a purely non-linear phenomenon!!!

- Assume $f \in C^2$ and $f''>0$





$$u_0(x - t f'(u(x,t))) = u(x,t)$$

$$u_t = u_0' \cdot \left(-f'(u(x,t)) - t f''(u(x,t)) \cdot u_t\right)$$

$$u_t(1 + t f'' u_0') = -u_0' f'$$

$$u_t = -\frac{u_0' f'}{1 + t f'' u_0'}$$

Analogously, $u_x = \dfrac{u_0'}{1 + t f'' u_0'}$

If $u_0' \geq 0$ (and $f''>0$) u_t and u_x stay bounded.

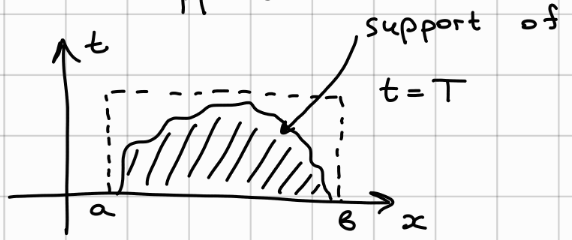
If $u_0' < 0$, then u_t and u_x become unbounded as $1 + t f'' u_0'$ tends to 0.

So we need a notion of *weak* solution!

Weak solutions to conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & (1) \\ u\big|_{t=0} = u_0(x) \end{cases}$$

Let u be a classical solution and $\varphi \in C^1$ with compact support:



$\operatorname{supp}(\varphi) \subset D = [a,b]\times[0,T]$

that is φ is zero at $x=a$, $x=b$, $t=T$

Multiply (1) by φ and integrate over $\mathbb{R}\times\mathbb{R}_+$

$$\iint_{t>0} (u_t + f(u)_x)\,\varphi \, dx\, dt = \iint_D (u_t + f(u)_x)\,\varphi\, dx\, dt =$$

$$= \int_a^b\int_0^T (u_t + f(u)_x)\,\varphi\, dx\, dt = \int_a^b u\cdot\varphi\Big|_0^{\cancelto{0}{T}} dx - \int_a^b\int_0^T u\cdot\varphi_t\, dx$$

$$+ \int_0^T \cancelto{0}{f(u)\cdot\varphi\Big|_a^b} dt - \int_0^T\int_a^b f(u)\cdot\varphi_x\, dx\, dt =$$

$$= -\int_a^b u_0(x)\varphi(x)\,dx - \int_a^b\int_0^T (u\varphi_t + f(u)\varphi_x)\,dx\,dt$$

$$\Rightarrow \quad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0(x)\varphi(x)\,dx = 0 \qquad (2)$$

$u \in C^1$ and satisfies (1) $\Rightarrow$ u satisfies (2)

But in (2) u not necessarily needs to be C^1!

It can be measurable / bounded.

Definition: A bounded measurable function $u(x,t)$ is called a weak solution of IVP:

$$u_t + f(u)_x = 0 \;, \quad u\big|_{t=0} = u_0(x)$$

$\uparrow$ bounded / meas.

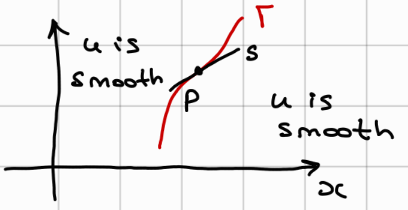
provided that

$$(2) \qquad \iint_{t\geq 0} (u\varphi_t + f(u)\varphi_x)\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0$$

for all $\varphi \in C^1_0$ (φ is C^1 with compact supp)

Rmk: it is clear that if u is in fact C^1, then the original eq. is true: $u_t + f(u)_x = 0$

Lemma (Rankine-Hugoniot condition)



Let Γ be a smooth curve across which u has a jump discontinuity. Take $P \in \Gamma$ and

$u_\ell = \lim_{(x,t)\to P} u$ from "the left"

$u_r = \lim_{(x,t)\to P} u$ from "the right"

Let the tangent line of Γ at P have the slope

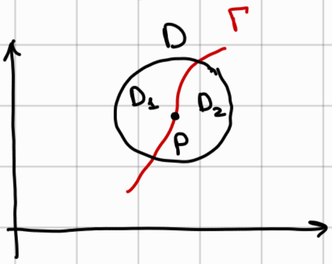
$s = \dfrac{dx}{dt}$. Then: (3) $\boxed{s\cdot(u_\ell - u_r) = f(u_\ell) - f(u_r)}$

Often a jump across the shock is denoted:

$[g(u)] = g(u_\ell) - g(u_r)$, thus we have $s\,[u] = [f]$

This is called the Rankine-Hugoniot condition

Proof :



Let D be a small ball centered at P and let Γ devide D into two regions D_1 and D_2 (see fig)

Let $\varphi \in C_0^1$ on D and consider

$$0 = \iint_D \left(u\varphi_t + f(u)\varphi_x\right) dx\,dt = \iint_{D_1} + \iint_{D_2}$$

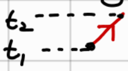
Divergence theorem :
(Green-Gauss theorem)

$$\int P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x \,dx\,dt \overset{\uparrow}{=} \iint_{D_1} (u\varphi)_t + \left(f(u)\varphi\right)_x \,dx\,dt =$$

as $u \in C^1(D_1)$ and $u_t + f(u)_x = 0$

$$= \int f(u)\varphi\,dt - u\varphi\,dx = \int f(u)\varphi\,dt - u\varphi\,dx =$$



$$= \int_{t_1}^{t_2} \left[f(u_e)\,\varphi(u_e) - u_e\cdot\varphi(u_e)\cdot s\right] dt$$

Similarly,

$$\iint_{D_1} u\varphi_t + f(u)\varphi_x \,dx\,dt = -\int_{t_1}^{t_2} \left(f(u_r) - su_r\right)\varphi(u_r)\,dt$$

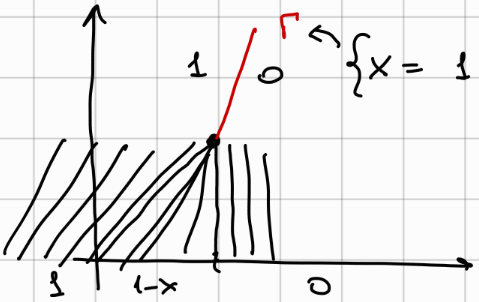
minus because of orientation of ∂D_2 :

Combining together:

$$0 = \int_{t_1}^{t_2} \left([f] - s[u]\right)\varphi(u_e)\,dt$$

Since φ was arbitrary, we get $[f] - s[u] = 0$ relation (3). ∎

Example : Burgers eq :



$$\left\{x = 1 + \frac{t-1}{2}\right\}$$

u_e-left state

u_r-right state

$$s = \frac{\left[\frac{u^2}{2}\right]\Big|_0^1}{[u]\Big|_0^1} = \frac{1}{2}$$

in general $s = \dfrac{u_e + u_r}{2}$

Lecture 7 Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u\big|_{t=0} = u_0(x) \end{cases}$ $(*)$

$u : \mathbb{R} \overset{\times \mathbb{R}_+}{\to} \mathbb{R}$ - bounded, measurable

$f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f'' > 0$ on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} (u \varphi_t + f(u) \varphi_x) \, dx \, dt + \int_{t=0} u_0 \varphi \, dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

We want to prove theorems on $\exists$, ! and asymptotic behavior of solutions to $(*)$. From exercise session 1 we remember that we need some extra conditions for this

$\boxed{\text{Thm1}}$ $(\exists)$:

Let $u_0 \in L_\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$, $f'' > 0$ on $\{u : |u| \leq \|u_0\|_\infty\}$

Then there exists a solution withe the following properties :

ⓐ $|u(x,t)| \leq \|u_0\|_\infty = M$, $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

ⓑ $\exists E > 0$ (which depends on M, $\mu = \min\{f''(u) : |u| \leq M_0$ and $A = \max\{|f'(x)| : |u| \leq \|u_0\|\}$) ~~such~~

such that $\forall a > 0, t > 0, x \in \mathbb{R}$

$$\frac{u(x+a, t) - u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

ⓒ u is stable and depends continuously on u_0: if $v_0 \in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \leq \|u_0\|_\infty$ and v is the corresponding constructed solution of $(*)$ with initial data v_0, then for $\forall x_1, x_2 \in$ with $x_1 < x_2$ and $\forall t > 0$

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)| \, dk \leq \int_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)| \, dx .$$

~~Remarks : 1)~~

$\boxed{\text{Thm2}}$ (!):

Let $f \in C^2$, $f'' > 0$ and let u and v be 2 solutions of $(*)$ satisfying condition (E). Then $u = v$ almost everywhere in $t > 0$.

Rmk : we call the solution from Thm 1 (that is satisf. (E may be there exist more solutions which do not satisfy cond. (E) or (c)

2) property (a) is not valid for systems! Sup-norm of solution can increase! It is non-trivial to prove the bounds on the sup-norm.

3) Cond. **(E)** implies some regularity : u is of locally bounded total variation (for $\forall t$ as a function of x)

Indeed, let c_1 be a constant such that $c_1 > \frac{E}{t}$ and let $v = u - c_1 x$. Then

$$v(x+a,t) - v(x,t) = u(x+a,t) - u(x,t) - c_1 a < a\left(\frac{E}{t} - c_1\right) < c$$

Thus, v is a non-decreasing function, and v is a function of local bounded total variation.

Since $c_1 x$ is also of bounded total variation, then u is of local bounded total variation.

($\Rightarrow$ countable number of jump discontinuities)

4) finite speed of propagation:

$$v = v_0 \equiv 0 \quad \Rightarrow \quad \int_{x_1}^{x_2} |u(x,t)|\, dx \le \int_{x_1 - At}^{x_2 + At} |u_0(x)| dx$$

Before proving thm 1 and 2 let us understand better condition (E). Let us give some equivalent formulations and interpetations.

<u>Lemma:</u> ⓐ A smooth solution $u(x,t)$ $(\forall t)$ satisfies condition (E)

ⓑ If u has a discontinuity at point x_0 : (but is smooth to the left and to the right of x_0) and $\lim\limits_{x \to x_0 - 0} u(x,t) = u_L$ and $\lim\limits_{x \to x_0 + 0} u(x,t) = u_R$ and satisfies condition (E) $\Rightarrow u_L > u_R$.

(discontinuities can be only down).

Proof:

⊳ ⓐ Indeed, let us write :

$$u(x,t) = u_0(\, x - t\, f'(u(x,t)))$$

$$u_x = u_0' \cdot (1 - t\, f''\, u_x) \Rightarrow u_x = \frac{u_0'}{1 + t f'' u_0'}$$

If u is smooth for $\forall t > 0$, then $u_0' > 0$.

Then $u_x \le \frac{u_0'}{t f'' u_0'} = \frac{E}{t}$ for $E = \frac{1}{\inf f''}$.

Using Lagrange theorem: $\dfrac{u(x+a,t)-u(x,t)}{a} = u_x(\xi,t)$

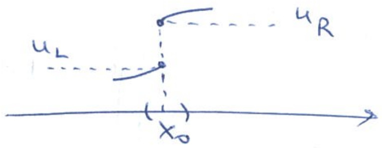
for some $\xi \in [x, x+a)$, and (a) is proved

ⓑ Either $u_L > u_R$ or $u_L < u_R$ (the case $u_L = u_R$ is not a discontinuity).

- For $u_L < u_R$ the converse of cond. Ⓔ is true:

$\forall E > 0 \ \exists x, a > 0, t$: $\dfrac{u(x+a)-u(x)}{a} > \dfrac{E}{t}$.

Indeed, fix E and take small enough neighbourhood of x_0 such that



- for $x \in (x_0 - \delta, x_0)$ $|u - u_L| \leq \varepsilon = \dfrac{u_R - u_L}{4}$
- for $x \in (x_0, x_0 + \delta)$ $|u - u_R| \leq \varepsilon = \dfrac{u_R - u_L}{4}$.

This means that for $\forall x_1 \in (x_0 - \delta, x_0)$ and $x_2 \in (x_0, x_0 + \delta)$ $u(x_2) - u(x_1) \geq \dfrac{u_R - u_L}{2}$.

Fix t and take $\overset{\text{small}}{\vee}$ a:

$x_2 - x_1 = a$

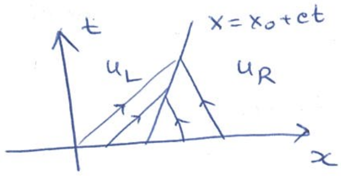
$x_1 \in (x_0 - \delta, x_0)$

$x_2 \in (x_0, x_0 + \delta)$

$$\frac{u(x_2)-u(x_1)}{a} \ [illegible] .$$
$$\geq$$
$$\frac{u_R - u_L}{2a} = \frac{E}{t}.$$

- For $u_L > u_R$ $\dfrac{u(x+a)-u(x)}{a} \leq 0$, thus $\forall E > 0$ is ok ■

Lemma 2 (Remark): u satisfies condition (E) and is a shock wave solution $u = \begin{cases} u_L, & x < ct \\ u_R, & x > ct \end{cases}$



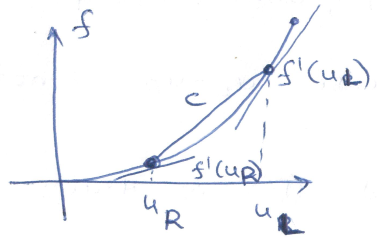
then $f'(u_L) > c > f'(u_R)$ (Lax conditio

"Characteristics come to the line o discontinuity"

The converse situation in some sense corresponds to the case  where "the information" appears on the discontinuity,- which is not

We will generalize the Lax condition to the case of systems.

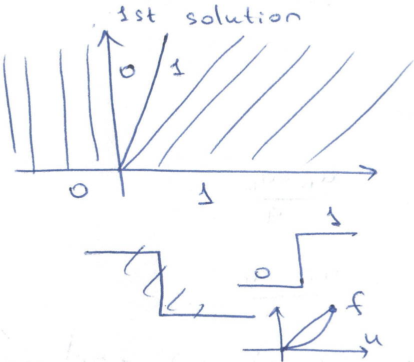
Indeed, $f''>0 \Rightarrow$ (see picture)

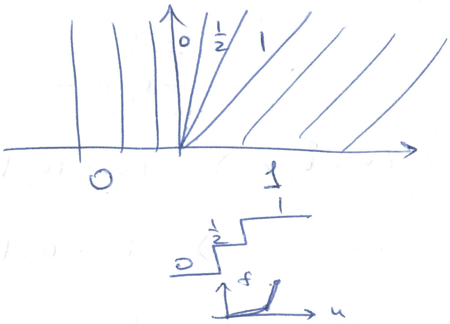


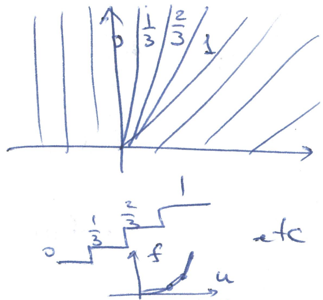
$$f'(u_L) > c = \frac{f(u_L)-f(u_R)}{u_L-u_R} > f'(u_R)$$

Remark (on Liu criterion) "internal stability of shock"

Remember the situation with Burgers equation:



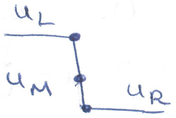




In some sense if we divide the shock into "2 smaller" shocks, they could have a tendency of eigher gluing into 1 shock (some kind of stabilite or going further one from another (instability)

Condition (E) $\Rightarrow$ this kind of internal stability of a shock, more precisely the inequalities

$$c(u_L,u_R) = \frac{f(u_R)-f(u_L)}{u_R-u_L} \leq c(u_L,u_M) = \frac{f(u_L)-f(u_M)}{u_L-u_M}$$



$$\geq \quad c(u_M,u_R) = \frac{f(u_R)-f(u_M)}{u_R-u_M} \qquad \forall u_M \in (\min(u_L, \; \max(u_L,u$$

If $\left[\begin{array}{l} u_M \to u_L \\ u_M \to u_R \end{array}\right.$ we have Lax condition.

Vanishing viscosity criterion for shock waves.

- We think of equation $u_t + (f(u))_x = 0$ as a first approximation to the following parabolic eq

$$u_t + (f(u))_x = \underbrace{\varepsilon u_{xx}}_{\text{small regularizing term}}, \quad \varepsilon > 0 \qquad (P)$$

Rmk 1: it is well-known (and we see in future when dealing with reaction-diffusion equations) that solutions of (P) are very regular (no shocks)

Rmk 2: equation (P) is a combination of 2 "opposite" effects

→ $u_t + (f(u))_x = 0$ ⇝ creates shocks: 

→ $u_t = \varepsilon u_{xx}$ ⇝ "smooths" : 

As a consequence of this confrontation there exist very special solutions, called travelling waves (TW) such that:

$$u(x,t) = v(x-ct)$$ 

for $c \in \mathbb{R}$ and v - some smooth profile.

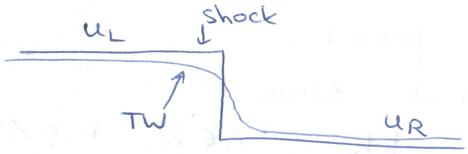
They look like "smoothed" shocks!!!

This motivates the following definition:

Def 1 (vanishing viscosity criterion for shock waves):

A shock wave is an entropy solution if is a limit in L^1_{loc} of a travelling wave solution of (P) as $\varepsilon \to 0$.

$f \in C^2$, $f'' > 0$



Lemma: a shock wave is an entropy solution in sense of def 1, iff $u_L > u_R$.

Proof: Let's look for travelling wave solutions for eq. (P): $v\left(\frac{x-ct}{\varepsilon}\right)$: $v(-\infty) = u_L$, $v(+\infty) = u$

$$-cv' + (f(v))' = \varepsilon v''$$

Integrate $\int_{-\infty}^{+\infty}$: $-c(u_R - u_L) + f(u_R) - f(u_L) =$

Interesting feature: it is exactly RH condition

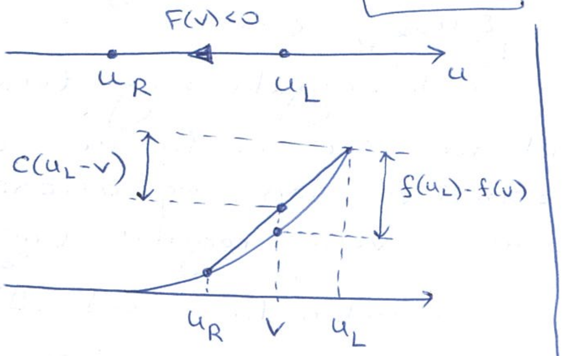
OK, let us integrate $\int_{-\infty}^{\xi}$: $-c(v(\xi) - u_L) + (f(v(\xi)) - f(u$
$= v'(\xi)$

ODE) $v' = f(v) - f(u_L) - c(v - u_L) =: F(v)$

Note that RHS $F(u_L) = 0$ and $F(u_R) = 0$ (due to RH!)

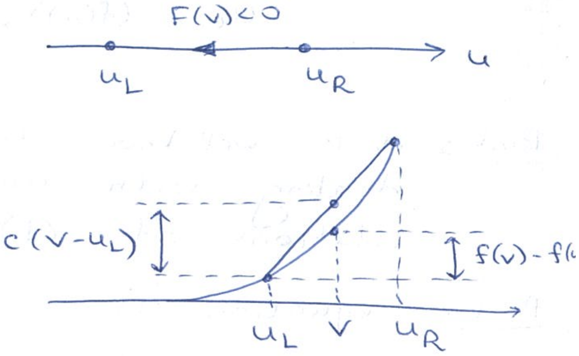
Thus u_L and u_R are two fixed points of this ODE

Consider 2 cases: $\boxed{u_L > u_R}$ and $\boxed{u_L < u_R}$.



In this case: $F(v) < 0$
$\forall v \in (u_R, u_L)$

And there exists a solution v of ODE:
$v(-\infty) = u_L$; $v(+\infty) = u_R$



In this case: $F(v) < 0$
$\forall v \in (u_L, u_R)$

And there DOES NOT exist a solution v of ODE
$v(-\infty) = u_L$; $v(+\infty) = u_R$

■

Lecture 8: Scalar conservation law: $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u: \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable
  $f: \mathbb{R}\to\mathbb{R}$, $f\in C^2$, $f''>0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense:

$$\iint_{t>0} \left[u\varphi_t + f(u)\varphi_x\right] dx\,dt + \int_{t=0} u_0\varphi\, dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

Define $M := \|u_0\|_\infty$, $A := \max\limits_{|u|\le M} |f'(u)|$, $\mu := \min\limits_{|u|\le M} f''(u)$

Today we will start proving theorem on existence.

Thm 1 $(\exists)$: Let $u_0 \in L_\infty(\mathbb{R})$; $f\in C^2(\mathbb{R})$, $f''>0$ on $\{u: |u|\le M\}$

There exists a solution with the following properties

ⓐ $|u(x,t)| \le M$, $(x,t)\in\mathbb{R}\times\mathbb{R}_+$

ⓑ $\exists E = E(M,\mu,A)>0$ such that $\forall a>0$, $\forall t>0$, $x\in\mathbb{R}$

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{E}{t} \qquad (E)$$

"entropy" cond.

ⓒ u is stable and depends continuously on u_0: if $v_0\in L_\infty(\mathbb{R})$ with $\|v_0\|_\infty \le \|u_0\|_\infty$ and v is the corresponding constructed solution of $(*)$ with initial data v_0, then for $\forall x_1,x_2\in\mathbb{R}$ with $x_1<x_2$ and $t>0$

$$\int_{x_1}^{x_2} |u(x,t)-v(x,t)|\,dx \le \int_{x_1-At}^{x_2+At} |u_0(x)-v_0(x)|\,dx \qquad (S)$$

"stability"

How to prove this theorem?

There exist (at least) 5 approaches:

(a) Calculus of variations and Hamilton-Jacobi theory

(b) Vanishing viscosity method

(c) Non-linear semigroup theory

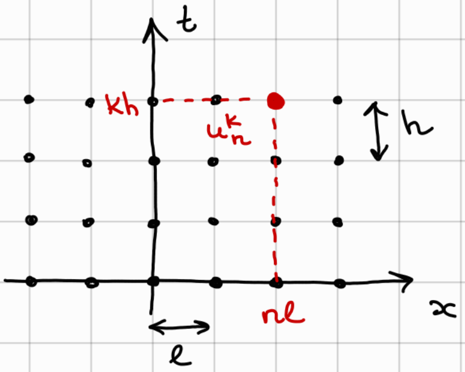
(d) Method of characteristics

(e) Finite - difference method

We will follow Smoller (Chapter 16) and use (e).

Here is the scheme of the proof :

Step 1 : discretization in space and time



$x_n = n\ell$, $n \in \mathbb{Z}$ $\ell = \Delta x > 0$

$t_k = kh$, $k \in \mathbb{N} \cup \{0\}$ $h = \Delta t > 0$

$u_n^k = u(n\ell, kh)$

Consider a finite-difference (explicit) scheme:

$$(D) \quad \boxed{u_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell} \cdot \left(f(u_{n+1}^k) - f(u_{n-1}^k) \right)}$$

$u_n^0 = u_0(n\ell)$, $n \in \mathbb{Z}$

In what follows we will always assume :

$$\boxed{\frac{Ah}{\ell} \le 1} \qquad (\text{CFL condition})$$

Courant-Friedrichs-Lewy

It is important for the stability of the numerical scheme and tells that the time step h should be small enough.

First, we will formulate and prove some properties of solutions u_n^k of a discrete eq. (D) :

(1a) solution exists (evident!)

(1b) if $|u_n^0| \le M$, then $|u_n^k| \le M$ $\forall k \in \mathbb{N}$

(boundedness)

(1c) $\exists E = E(M, A, \mu) > 0 :$ $\dfrac{u_n^k - u_{n-2}^k}{2\ell} \leq \dfrac{E}{kh}$ (E-disc)

discrete entropy condition

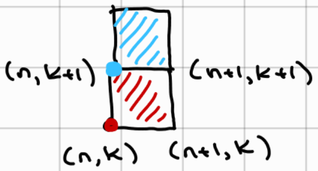
NB: the discrete entropy condition is a natural consequence of a finite difference approximation (D).

(1d) local bounded variation : $\forall X > 0$ and $Kh > \alpha > 0$
$\exists\, c(X, \alpha)$ (but independent of h and ℓ):

$$\sum_{|n| \leq X/\ell} |u_{n+2}^k - u_n^k| \leq C$$ and some other...

Step 2: We will prove convergence as $h, \ell \to 0$.

Consider $U_{h,\ell}(x, t) = u_n^k$ if
$n\ell \leq x \leq (n+1)\ell$
$kh \leq t \leq (k+1)h$



We will prove that there exists subsequence U_{h_i,ℓ_i} of $U_{h,\ell}$ such that $U_{h_i,\ell_i} \to u(x,t)$ - some measurable function

Step 3: We will prove that this limiting function satisfies integral equality (**) and all properties of theorem on $\exists$.

Proof of the theorem 1.

Lemma 1 (boundedness of u_n^k) : $|u_n^k| \leq M$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$

This is an exercise 2 from list 3.

Lemma 2 (discrete entropy condition)

If $c = \min\left(\frac{\mu}{2}, \frac{A}{4M}\right)$, then

$$\frac{u_n^k - u_{n-2}^k}{2\ell} \leq \frac{E}{kh} \quad \text{where } E = \frac{1}{c}.$$

Proof:

Γ ▶ Let $z_n^k = \dfrac{u_n^k - u_{n-2}^k}{2\ell}$ and first let us prove some recurrent relation for z_n^{k+1} of the form

" $z_n^{k+1} = A\, z_{n+1}^k + B\, z_{n-1}^k + C$ "

$$z_n^{k+1} = \frac{1}{2}\left[z_{n+1}^k + z_{n-1}^k\right] - \frac{h}{(2\ell)^2}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$
$$+ \frac{h}{(2\ell)^2}\left(f(u_{n-1}^k) - f(u_{n-3}^k)\right)$$

Notice that due to $f \in C^2$ we can write

$$f(u_{n+1}^k) = f(u_{n-1}^k) + f'(u_{n-1}^k)\left(u_{n+1}^k - u_{n-1}^k\right) + f''(\theta_1)\frac{\left(u_{n+1}^k - u_{n-1}^k\right)^2}{2}$$

for some θ_1 between u_{n+1}^k and u_{n-1}^k

$$= f(u_{n-1}^k) + f'(u_{n-1}^k)\cdot 2\ell\cdot z_{n+1}^k + f''(\theta_1)\cdot\frac{(2\ell)^2}{2}\left(z_{n+1}^k\right)^2$$

Analogously,

$$f(u_{n-3}^k) = f(u_{n-1}^k) - f'(u_{n-1}^k)\cdot 2\ell\cdot z_{n-1}^k + f''(\theta_2)\frac{(2\ell)^2}{2}\left(z_{n-1}^k\right)^2$$

Thus,

$$z_n^{k+1} = z_{n+1}^k\cdot\Big[\overbrace{\frac{1}{2} - \frac{h}{2\ell}f'(u_{n-1}^k)}^{A}\Big] + z_{n-1}^k\Big[\overbrace{\frac{1}{2} + \frac{h}{2\ell}f'(u_{n-1}^k)}^{B}\Big]$$
$$-\frac{h}{2}\cdot\left[f''(\theta_1)\cdot\left(z_{n+1}^k\right)^2 + f''(\theta_2)\cdot\left(z_{n-1}^k\right)^2\right]$$

Note that $A + B = 1$ and $A, B \geq 0$

Define $\tilde{z}_n^k = \max\{z_{n-1}^k, z_{n+1}^k, 0\}$

If $\tilde{z}_n^k = 0 \ \forall n$, then (E-disc) is true since
$$z_n^k \leq \tilde{z}_n^k = 0 \leq \frac{E}{kh} \quad \forall E > 0.$$

Thus, w.l.o.g. we can assume $\tilde{z}_n^k \neq 0$. Suppose $\tilde{z}_n^k = z_{n+1}^k$ (the other case is similar)

$$z_n^{k+1} \leq z_{n+1}^k\cdot\left[A + B\right] - h\frac{\mu}{2}\left(z_{n+1}^k\right)^2 \leq$$
$$\leq \tilde{z}_n^k + h c\left(\tilde{z}_n^k\right)^2 .$$

Notice that

$$z_n^k \leq |z_n^k| \leq \frac{M}{\ell} \underset{\substack{\uparrow \\ \text{CFL}}}{\leq} \frac{M}{Ah} \underset{\substack{\uparrow \\ c \,\leq\, \frac{A}{4M}}}{\leq} \frac{M}{h}\cdot\frac{1}{4MC} = \frac{1}{4ch}$$

Let $M^k = \max\limits_{n\in\mathbb{Z}}\{\tilde{z}_n^k\} \geq 0$.

Let $\varphi(y) = y - c\cdot h\cdot y^2$. Since $\varphi' = 1 - 2chy$, φ is increasing if $y \le \frac{1}{2ch}$. But we have $\tilde{z}_n^k \le M^k \le \frac{1}{4ch} < \frac{1}{2ch}$.

So that $\phi(\tilde{z}_n^k) \le \phi(M^k)$ and we have

$$\tilde{z}_n^k - ch(\tilde{z}_n^k)^2 \le M^k - ch(M^k)^2$$

Thus, $z_n^{k+1} \le M^k - ch(M^k)^2 \quad \forall n \in \mathbb{Z}$.

It follows that

$$\boxed{M^{k+1} \le M^k - ch(M^k)^2} \qquad (M)$$

Claim: $M^k \le \frac{1}{chk + 1/M^0}$.

Suppose we have proven claim. Let us see how it helps to prove lemma 2. Indeed,

$$z_n^k \le M^k \le \frac{1}{chk + 1/M^0} \le \frac{1}{chk} = \frac{E}{hk}, \quad E = \frac{1}{c}.$$

Proof of claim: first – intuition why such estimate could be true

Inequality (M) for M^k is a discrete analog of ODinEquality: $\varphi' \le -ch\varphi^2$

if it was an equality $\varphi' = -ch\varphi^2$, then the solution is:

$$\frac{d\varphi}{\varphi^2} = -ch\,dt$$

$$-\frac{1}{\varphi} = -cht + C_1$$

$\varphi(t) = \frac{1}{cht - C_1}$ and with ic $\varphi(0) = \varphi_0$

we will have $\varphi(t) = \frac{1}{cht + 1/\varphi_0}$.

So one can try to prove $\varphi(t) \le \frac{1}{cht + 1/\varphi_0}$.

Second, let us make the formal proof.

We will do it by induction.

Base : $k=0$: – clear : $M^0 \leq \frac{1}{1/\mu^0} = M^0$.

$k>0$: suppose that

$$M^k \leq \frac{1}{chk + 1/\mu^0}$$

and we want to prove that

$$M^{k+1} \leq \frac{1}{ch(k+1) + 1/\mu^0}$$

We have : $\frac{1}{M^k} \geq chk + \frac{1}{M^0}$, so

$$1 - chM^k \geq 1 - chk\, M^k \geq \frac{M^k}{M^0} \geq 0.$$

Thus $1 - (ch\, M^k)^2 \geq 0$.

We have $M^{k+1} \leq M^k \left(1 - \overset{\overset{0}{\wedge}}{c}h M^k \right)$, so that

$$\frac{M^{k+1}}{1 - ch\, M^k} \leq M^k \leq \frac{M^k}{1 - (ch\, M^k)^2}$$

and thus $M^{k+1} \leq \frac{M^k}{1 + ch\, M^k} = \frac{1}{ch + 1/\mu^k} \leq$

$$\leq \frac{1}{ch(k+1) + 1/\mu^0} \qquad \text{q.e.d.} \quad ■$$

└

Lemma (space estimate) : For any $X>0$ and $kh \geq \alpha > 0$, there is a constant $C = C(X, \alpha, M)$ (but independent on h, e) such that:

$$\sum_{|n| \leq X/e} |u_{n+2}^k - u_n^k| \leq C$$

Proof :

┌▶ Set $v_n^k = u_n^k - c_1 n e$, where c_1 is chosen so large that $E/\alpha < c_1$. Then

$$v_{n+2}^k - v_n^k = u_{n+2}^k - u_n^k - 2c_1 e \leq \frac{2eE}{kh} - 2c_1 e \leq$$

$$\leq 2e \left(\frac{E}{\alpha} - c_1 \right) < 0 \text{ , so } v_n^k \text{ is decreas. in } n$$

Thus

$$\sum_{|n| \leq X/e} |u_{n+2}^k - u_n^k| \leq \sum |v_{n+2}^k - v_n^k| + \sum 2c_1 e =$$

$$= \underbrace{-\sum (v_{n+2}^k - v_n^k)}_{\text{telescopic sum}} + 2c_1 e \left(\frac{2X}{e} + 1 \right) \leq 4M + 2c_1 X + c_2 X$$

$$\leq 4(M + c_1 X)$$

└ q.e.d. ■

Lecture 9 : We continue proving theorem on existance of entropy solution for scalar cons law.

Lemma 4 (time estimate – u_n^k are L^1 locally Lipshits in k)

If $h/\ell \geq \delta > 0$ and $h, \ell \leq 1$, then exists $L > 0$ (independent of h, ℓ) such that

if $k > p$, where $(k-p)$ is even and $ph \geq \alpha > 0$, then

$$\sum_{|n| \leq X/\ell} |u_n^k - u_n^p| \, \ell \leq L \, (k-p) \, h$$

A similar estimate holds if $(k-p)$ is odd.

Proof:

◀ Let us express u_n^k in terms of u_n^p where $(k-p)$ is even.

$$u_n^k = \frac{1}{2}\left(u_{n+1}^{k-1} + u_{n-1}^{k-1}\right) - \frac{h}{2\ell} f'(\Theta) \left(u_{n+1}^{k-1} - u_{n-1}^{k-1}\right) =$$

$$= u_{n+1}^{k-1}\left(\frac{1}{2} - \frac{h}{2\ell} f'(\Theta)\right) + u_{n-1}^{k-1}\left(\frac{1}{2} + \frac{h}{2\ell} f'(\Theta)\right)$$

or $u_n^k = a_{n+1}^{k-1} u_{n+1}^{k-1} + a_{n+1}^{k-1} u_{n+1}^{k-1}$, where

$a_{n+1}^{k-1} + a_{n-1}^{k-1} = 1$ and $a_{n+1}^{k-1}, a_{n-1}^{k-1} \geq 0$.

Applying this to u_{n-1}^k and u_{n+1}^k gives a formula:

$$u_n^{k+1} = A u_{n+2}^{k-1} + B u_n^{k-1} + C u_{n-2}^{k-1}$$

where $A, B, C \geq 0$, $A + B + C = 1$.

Hence, $|u_n^{k+1} - u_n^{k-1}| \leq A |u_{n+2}^{k-1} - u_n^{k-1}| + C |u_{n-2}^{k-1} - u_n^{k-1}|$

Multiplying this by $\Delta x = \ell$ and summing, we get:

$$\sum_{|n| \leq X/\ell} |u_n^{k+1} - u_n^{k-1}| \, \Delta x \underset{\substack{\uparrow \\ \text{lemma 3}}}{\leq} c \, \Delta x$$

Now if $(k-p)$ is even, we can do this operation several times and using the triangle inequality, we get:

$$\sum_{|n| \leq X/\ell} |u_n^k - u_n^p| \, \Delta x \leq \sum_{i=p}^{k-2} \sum_{|n| \leq X/\ell} |u_n^{i+2} - u_n^i| \, \Delta x \leq (k-p) c \, \Delta x \leq$$

$$\leq \frac{\Delta t}{\delta}(k-p)c = L(k-p)h \quad \text{for } L = \frac{c}{\delta}, \; h = \Delta t \; ■$$

⌊

Lemma 5 (stability): Let u_n^k and v_n^k be solutions to the finite-difference scheme (D) corresponding to the initial conditions u_n^0 and v_n^0, respectively, where

$$\sup_{n\in\mathbb{Z}} |u_n^0| \leq M \quad \text{and} \quad \sup_{n\in\mathbb{Z}} |v_n^0| \leq M$$

Then, if $k>0$,

$$\sum_{|n|\leq N} |u_n^k - v_n^k| \cdot \ell \leq \sum_{|n|\leq N+k} |u_n^0 - v_n^0| \cdot \ell$$

Proof:

⌈ ▶ $w_n^k = u_n^k - v_n^k$. From (D) we have

$$w_n^{k+1} = u_n^{k+1} - v_n^{k+1} = \frac{u_{n+1}^k + u_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(u_{n-1}^k)\right)$$

$$- \frac{v_{n+1}^k + v_{n-1}^k}{2} + \frac{h}{2\ell}\left(f(v_{n+1}^k) - f(v_{n-1}^k)\right) =$$

$$= \frac{w_{n+1}^k + w_{n-1}^k}{2} - \frac{h}{2\ell}\left(f(u_{n+1}^k) - f(v_{n+1}^k)\right)$$

$$+ \frac{h}{2\ell}\left(f(u_{n-1}^k) - f(v_{n-1}^k)\right) =$$

$$= w_{n+1}^k \underbrace{\overset{A_{n+1}^k \, =}{\left[\frac{1}{2} - \frac{h}{2\ell} f'(\Theta_1)\right]}}_{\geq 0 \text{ due to CFL}} + w_{n-1}^k \underbrace{\overset{= \, B_{n+1}^k}{\left[\frac{1}{2} + \frac{h}{2\ell} f'(\Theta_2)\right]}}_{\geq 0}$$

Now proceed by induction.

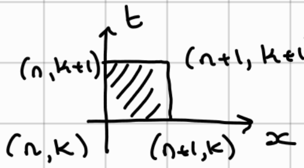
$$\sum_{|n|\leq N} |w_n^{k+1}| \leq \sum_{|n|\leq N} |w_{n+1}^k| \cdot A_{n+1}^k + \sum_{|n|\leq N} |w_{n-1}^k| \cdot B_{n+1}^k =$$

$$= \sum_{1-N}^{N+1} |w_m^k| \, A_m^k + \sum_{-1-N}^{N-1} |w_m^k| \cdot B_m^k \leq$$

$$\leq \sum_{|m|\leq N+1} |w_m^k| \, A_m^k + \sum_{|m|\leq N+1} |w_m^k| \cdot B_m^k \leq \sum_{|m|\leq N+1} |w_m^k| \; ■$$

q.e.d.

Step 2: Rather than define u_n^k in mesh points let us continue u_n^k as a piece wise constant function in the upper half plane.



$$\mathcal{U}_{h,\ell}(x,t) = u_n^k \quad \text{if} \quad n\ell \le x \le (n+1)\ell$$
$$kh \le t \le (k+1)h$$

So we have a family of functions $\{\mathcal{U}_{h,\ell}\}$ and would like to choose a convergent subsequence $\mathcal{U}_{h_i,\ell_i}$ as $h_i, \ell_i \to 0$ $i \to \infty$.

Lemma 6 (convergence: the set of functions $\{\mathcal{U}_{h,\ell}\}$ is compact in the topology of L_1-convergence on compacta)

There exists a subsequence $\{\mathcal{U}_{h_i,\ell_i}\}$ $i \in \mathbb{N}$ which converges to a measurable function $u(x,t)$ in the sense that for $\forall X > 0, t > 0, T > 0$ both

$$\int_{|x| \le X} |U_{h_i,\ell_i}(x,t) - u(x,t)| \, dx \to 0 \quad \text{as } h_i, \ell_i \to 0$$

and

$$\int_0^T \int_{|x| \le X} |U_{h_i \ell_i}(x,t) - u(x,t)| \, dx \, dt \to 0 \, .$$

Furthermore, the function $u(x,t)$ satisfies:

(a) $\sup\limits_{\substack{x \in \mathbb{R} \\ t > 0}} |u(x,t)| \le M$; (b) inequality (S) (stability)

Proof:

▶ First, take $t = \text{const}$ and consider $U_{h,\ell}(x,t)$ as functions of x. By Lemma 1 and Lemma 3 the set of functions $\{U_{h,\ell}\}$ is bounded and have uniformly bounded total variation on each bounded interval in x.

Helly's theorem (simple version):

A uniform bounded sequence of monotone, real functions admits a convergent subsequence.

Helly's theorem (generalized version):

A uniform bounded sequence of BV_{loc} (locally of bounded variation) real functions admits a convergent subsequence on every compact set.

Rmk: a function of BV_{loc} can be written as a sum of increasing and decreasing functions (on each compact interval). This is why the generalized version of the Helly's theorem is true.

So by Helly's theorem on each interval we have a convergent subsequence $U'_{h,\ell}$.

By a standard diagonal process we can construct a subsequence $\{U''_{h,\ell}\}$ from $\{U'_{h,\ell}\}$ which converges at every $x \in \mathbb{R}$ for this particular $t = const > 0$.

Second, take $\{t_m\}_{m=1}^{\infty}$ - a countable and dense subset of $(0,T)$, e.g. $\mathbb{Q} \cap (0,T)$.

For $t=t_1$ we have $\{U^1_{h_i,\ell_i}\}$ a convergent subsequence.

For $t=t_2$ take a convergent sub. $\{U^2_{h_i,\ell_i}\}$ from $\{U^1_{h_i,\ell_i}\}$

etc. So we have:

$$\begin{array}{lllll}
t=t_1: & \boxed{U^1_{h_1,\ell_1}} & U^1_{h_2\ell_2} & U^1_{h_3,\ell_3} & U^1_{h_4,\ell_4} \\
t=t_2: & U^2_{h_1,\ell_1} & \boxed{U^2_{h_2,\ell_2}} & U^2_{h_3,\ell_3} & U^2_{h_4,\ell_4} \\
t=t_3: & U^3_{h_1,\ell_1} & U^3_{h_2,\ell_2} & \boxed{U^3_{h_3,\ell_3}} & U^3_{h_4,\ell_4} \\
t=t_4: & U^4_{h_1,\ell_1} & U^4_{h_2,\ell_2} & U^4_{h_3,\ell_3} & \boxed{U^4_{h_4,\ell_4}} \\
 & & & & \ldots
\end{array}$$

By a standard diagonal process, we can choose a subsequence U_{h_i,ℓ_i} which converges for all $\{t_m\}_{m=1}^{\infty}$ and all $x \in \mathbb{R}$.

Third, we want to show that there is a convergence for all $t \in (0,T)$. So that in the limit we indeed obtain a function defined in the strip $0<t\leq T$

Let $U_i = U_{l_i, h_i}$ and we want to show that

$$I_{i,j} = \int_{-X}^{X} |U_i(x,t) - U_j(x,t)|\, dx \longrightarrow 0 \quad \forall i,j \to \infty$$

i.e. that $\{U_i\}$ is a Cauchy sequence in $L_1(|x| \leq X)$

For $t \in (0,T)$ we find a subsequence $\{t_{m_s}\} \subset \{t_m\}$ such that $t_{m_s} \to t$ as $s \to \infty$. Let $\tau_s = t_{m_s}$. Then

$$I_{ij}(t) \leq \int_{-X}^{X} |U_i(x,t) - U_i(x,\tau_s)|\, dx + \int_{-X}^{X} |U_i(x,\tau_s) - U_j(x,\tau_s)|$$
$$+ \int_{-X}^{X} |U_j(x,t) - U_j(x,\tau_s)|\, dx =: I_1 + I_2 + I_3$$

For $t = \tau_s$ we have a convergence of U_i, thus for s large enough we have $I_2 < \varepsilon/3$

Let's estimate I_1:

$$I_1 = \int_{-X}^{X} |U_i(x, [\tfrac{t}{h_i}]h_i) - U_i(x, [\tfrac{\tau_s}{h_i}]h_i)| =$$
$$= \sum_{|n| < \frac{X}{l_i}+1} \int_{nl_i}^{(n+1)l_i} |U_i(x, [\tfrac{t}{h_i}]h_i) - U_i(x, [\tfrac{\tau_s}{h_i}]h_i)|\, dx =$$
$$= \sum_{|n| < \frac{X}{l_i}+1} |u_n^{[t/h_i]} - u_n^{[\tau_s/h_i]}|\, l_i \underset{\substack{\uparrow \\ \text{Lemma 4}}}{\leq} L h_i |[\tfrac{t}{h_i}] - [\tfrac{\tau_s}{h_i}]|$$

$$\leq L|t - \tau_s| < \frac{\varepsilon}{3} \quad \text{for } s \text{ large enough.}$$

Analogously, $I_3 < \dfrac{\varepsilon}{3}$. Thus $I_{ij} \leq 3 \cdot \dfrac{\varepsilon}{3} = \varepsilon$.

We have proved pointwise limit for every $t \in (0,T)$, that is $\exists\ u(x,t) \in L_1(|x| \leq X)$ (in part, measurable)

Fourth, let us show that $I_{ij} \to 0$ uniformly in t, $0 < \tau \le t \le T$. Indeed, fix $\varepsilon > 0$. Choose finite subset $\mathcal{F} \subset \{t_m\}$ such that if $0 \le t \le T$ there is a $t_m \in \mathcal{F}$ such that $L(t - t_m) < \frac{\varepsilon}{3}$. Then we choose i, j so large that $I_2 < \frac{\varepsilon}{3}$ for all $t_m \in \mathcal{F}$ (it is possible because $\mathcal{F}$ is finite) This reasoning gives us the desired uniformity in t

Fifth, using uniform convergence, we have

$$\forall \tau \in (0,T] \qquad \int_\tau^T I_{ij}\,dt \to 0 .$$

Now we write $\int_0^T = \int_0^\tau + \int_\tau^T$:

$$\int_0^T\int_{-X}^{X} |U_i - U_j|\,dx\,dt = \underbrace{\int_0^\tau\int_{-X}^{X} |U_i - U_j|}_{< \frac{\varepsilon}{2} \text{ if } 8MX\tau < \varepsilon} + \overbrace{\int_\tau^T\int_{-X}^{X} |U_i - U_j|}^{< \frac{\varepsilon}{2} \text{ for } i,j \text{ suffic. large}} < \varepsilon$$

That means

$$\int_0^T I_{ij}\,dt \longrightarrow 0 \quad \text{as} \quad i,j \to +\infty .$$

Sixth, since local convergence in L_1 implies pointwise convergence a.e. of a subsequence, we see

$|U_i| \le M \Rightarrow |u| \le M$

and Lemma 5 $\Rightarrow$ (S) ■

Step 3: Let us show that the limiting function $u(x,t)$, indeed, satisfies the properties from thm 1.

Lemma 7 (entropy inequality): u satisfies (E).

Proof:

⌈ ▶ It is sufficient to show that if $(x_1 - x_2) > 2l_i$ and $t > h_i$ then

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} < \frac{2E}{t-h_i}.$$

Let $x_1 > x_2$ and note that

$$U_i(x_j,t) = U_i\left(x_j - \eta_j,\ \left[\frac{t}{h_i}\right] h_i\right) \quad j=1,2$$

for some $0 \le \eta_j < l_j$. Thus,

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} = \frac{1}{x_1-x_2} \sum (u_n^k - u_{n-2}^k) \qquad \text{for } k = \left[\frac{t}{h_i}\right]$$

$\nearrow$ over all integers in the interval $[x_2-\eta_2, x_1-\eta_1]$

Using Lemma 2, we have

$$\frac{U_i(x_1,t) - U_i(x_2,t)}{x_1 - x_2} \le \frac{E(x_1-\eta_1-x_2+\eta_2)}{\left[\frac{t}{h_i}\right] h_i (x_1-x_2)} \le \frac{E(x_1-\eta_1-x_2+\eta_2)}{(t-h_i)(x_1-x_2)} =$$

$$= \frac{E}{t-h_i} + \frac{E\overbrace{(\eta_2-\eta_1)}^{<l_i}}{(t-h_i)\underbrace{(x_1-x_2)}_{>2l_i}} < \frac{2E}{t-h_i} \ ∎$$

⌊

Lecture 10 : Let's finish proving theorem on $\exists$ of entropy solution

Reminder : Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable

$f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f''>0$. As we will see it is enough to define f on the convex hull of values of u_0

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\,dx\,dt + \int_{t=0} u_0\varphi\,dx = 0 \qquad (**)$$

for every test function $\varphi \in C^1_0$.

Lemma 8 (last lemma)

Let V_i be a convergent subsequence from Lemma 6.

We know that $V_i \to u(x,t)$, $i \to +\infty$, and $\forall X \in \mathbb{R}$

$$\int_{-X}^{X} |V_i(x,0) - u_0(x)|\,dx \to 0.$$

Then u satisfies $(**)$, i.e. u is a weak solution of $(*)$.

Proof.

▶ Rewrite (D) in such a form:

$$\frac{u_n^{k+1}-u_n^k}{h} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2\ell^2}\cdot\frac{\ell^2}{h} + \frac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2\ell} = 0$$

Multiply this equality by $\varphi_n^k = \varphi(n\ell, kh)$ and get

$$\frac{\varphi_n^{k+1}u_n^{k+1} - \varphi_n^k u_n^k}{h} - u_n^{k+1}\frac{\varphi_n^{k+1}-\varphi_n^k}{h} + \frac{\ell^2}{h}\cdot u_n^k\cdot\frac{2\varphi_n^k - \varphi_{n+1}^k - \varphi_{n-1}^k}{\ell^2}$$

$$+ \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} + \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} +$$

$$+ \frac{\varphi_{n+1}^k f(u_{n+1}^k) - \varphi_{n-1}^k f(u_{n-1}^k)}{2\ell} - f(u_{n+1}^k)\frac{\varphi_{n+1}^k - \varphi_n^k}{2\ell} - f(u_{n-1}^k)\frac{\varphi_n^k - \varphi_{n-1}^k}{2\ell} = 0$$

Since $\varphi \in C^3_0$ has compact support, we may assume $\varphi_n^k = 0$ if $k \geq [\frac{T}{h}]$

Multiply this equality by hl and sum over $n \in \mathbb{Z}$, $k \in \mathbb{N} \cup \{0\}$.

- $\displaystyle\sum_{k,n} \frac{\varphi_n^{k+1} u_n^{k+1} - \varphi_n^k u_n^k}{h} = -\sum_n \varphi_n^0 u_n^0$ (telescopic sum)
- $\displaystyle\sum_{k,n} \frac{\varphi_{n+1}^k u_n^k - \varphi_n^k u_{n-1}^k}{2h} = 0$ and $\displaystyle\sum_{k,n} \frac{\varphi_{n-1}^k u_n^k - \varphi_n^k u_{n+1}^k}{2h} = 0$

Thus,

$$-h\sum_n \varphi_n^0 u_n^0 + hl\Big[\sum_{k,n}\Big[-u_n^{k+1}\frac{\varphi_n^{k+1}-\varphi_n^k}{h} - \frac{l^2}{2h}\overset{u_n^k}{\vee}\frac{\varphi_{n+1}^k+\varphi_{n-1}^k-2\varphi_n^k}{2l}\Big]$$
$$- \sum_{k,n} f(u_{n+1}^k)\frac{\varphi_{n+1}^k-\varphi_n^k}{2l} - \sum_{k,n} f(u_{n-1}^k)\frac{\varphi_n^k-\varphi_{n-1}^k}{2l}\Big] = 0$$

Instead of a sum for u_n^k we can write integral for $U_{h,l}$

$$-\int_{t=0} U_{h,l}\,\varphi + \delta_1 - \iint_{t\geq 0} U_{h,l}\,\varphi_t + \delta_2 - \frac{l^2}{2h}\iint_{t\geq 0} U_{h,l}\,\varphi_{xx}$$
$$+\delta_3 - \iint_{t\geq 0} f(U_{h,l})\,\varphi_x + \delta_4 = 0$$

where $\delta_i \to 0$ as $h, l \to 0$. Replace $U_{h,l}$ by U_i:

$$-\int_{t=0} U_i\,\varphi - \iint_{t\geq 0} U_i\,\varphi_t - \frac{l_i^2}{2h_i}\iint_{t\geq 0} U_i\,\varphi_{xx} - \iint_{t\geq 0} f(U_i)\,\varphi_x = \underset{\substack{\downarrow\, i\to\infty \\ 0}}{\delta(h_i,l_i)}$$

$l_i \to 0$, $\frac{l_i}{h_i}$ is bounded; $\frac{l_i^2}{h_i} \to 0$; $U_i \to u$ in L^1-loc

$$\Rightarrow \iint_{t\geq 0} U_i\,\varphi_t - \frac{l_i^2}{2h_i}\iint_{t\geq 0} U_i\,\varphi_{xx} \longrightarrow \iint_{t\geq 0} u\,\varphi_t$$

By choice of initial values: $\int_{t=0} U_i \varphi \to \int_{t=0} u_0 \varphi$

Also, $\left| \iint_{t\geq 0} \left(f(U_i) - f(u)\right) \varphi_x \right| \leq \|\varphi_x\|_\infty \iint_{D:\,\varphi\neq 0} | f(U_i) - f(u)|$

$$\leq \|\varphi_x\|_\infty \iint_{D:\,\varphi\neq 0} |f'(\xi)|\cdot|U_i-u| \to 0$$

And we have:
$$\iint_{t\geq 0} f(U_i)\varphi_x \to \iint_{t\geq 0} f(u)\varphi_x .$$

We have proved $(**)$ for $\forall\ \varphi \in C^3_0$.
$C^3_0 \subset C^1_0$ is a dense subset, then $(**)$ are also true for $\varphi \in C^1_0$. ■

Now, let's prove the theorem on uniqueness.

Thm 2 (!): Let $f \in C^2$, $f''>0$.
Let u, v be 2 solutions of $(**)$, satisfying entropy condition (E): $\exists E\ \forall a>0,\ t>0,\ x\in\mathbb{R}$

$$\frac{u(x+a)-u(x)}{a} < \frac{E}{t}. \qquad (E)$$

Then $u=v$ almost everywhere in $t>0$.

Rmk 1: we call such a solution – an entropy sol.

Rmk2: If we had a linear operator, then the main idea of the proof could be as follows (we will adapt this idea to non-linear)

Let H be a Hilbert space.

$A: H\to H$, $\eta(A) = \{g\in H:\ A(g)=0\}$ - null space

$R(A) = \{f\in H:\ \exists g\in H:\ A(g)=f\}$ - range of A

A^* is the adjacent operator:

$\langle Ax, y\rangle = \langle x, A^* y\rangle$

$R(A^*)$ is the orthogonal complement of $\eta(A)$

Fact: $R(A^*) \oplus \eta(A) = H$

The "bigger" is $R(A^*)$, the "smaller" is $\eta(A)$.
That means that if there exist sufficiently many solutions to the adjoint equation, then the null space of A is zero $\Rightarrow$ A has a unique solution !

If $Ax = Ay$ we can choose $w : A^* w = x - y$:

$$\|x-y\| = \langle x-y, x-y \rangle = \langle x-y, A^* w \rangle = \langle Ax - Ay, w \rangle = 0$$

$\Rightarrow x = y$ (idea of Holgrem ~ 1901)

But we have a nonlinear eq !

Let us adapt this idea.

Proof of thm 2.

▶ Let u, v be 2 solutions of (**).
In order to prove that $u = v$ a.e. in $t > 0$ it suffices to show that $\forall \varphi \in C_0^1$:

$$\iint_{t>0} (u-v)\varphi = 0 .$$

What we know? Let $\psi \in C_0^1$, then

(1) $$\iint_{t\geq 0} [u\psi_t + f(u)\psi_x]\,dx\,dt + \int_{t=0} u_0 \psi\, dx = 0$$

(2) $$\iint_{t\geq 0} [v\psi_t + f(v)\psi_x]\,dx\,dt + \int_{t=0} u_0 \psi\, dx = 0$$

Subtract (1) - (2) and we get:

$$\iint_{t\geq 0} (u-v)\left[\psi_t + \underbrace{\frac{f(u)-f(v)}{u-v}}_{=:F(x,t)} \cdot \psi_x\right] dx\,dt = 0$$

$$\iint_{t\geq 0} (u-v)\,[\psi_t + F\psi_x]\,dx\,dt = 0$$

?" $\varphi \in C_0^1$

Now if for $\forall\ \varphi \in C_0^1$ we could solve the linear (adjoint!) equation and have a solution $\psi \in C_0^1$, we could conclude that $u=v$ a.e.

However, there is an obstruction to this approach: "velocity field" F is not smooth (not even continuous), so it is not clear why solution $\psi \in C_0^1$.

To struggle this difficulty, one can approximate u and v by smooth functions and solve corresponding linear eqs:

$$(M) \qquad \psi_t^m + F_m \psi_x^m = \Phi, \quad F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m}$$

Then $\iint\limits_{t\geq 0} (u-v)\,\varphi = \iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$

$$= \underbrace{-\iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F \psi_x^m\right]}_{=0} + \iint\limits_{t\geq 0} (u-v)\left[\psi_t^m + F_m \psi_x^m\right] =$$

$$= \iint\limits_{t\geq 0} (u-v)\cdot\left[F_m - F\right]\cdot \psi_x^m$$

If $F_m \to F$ locally in L_1

ψ_x^m is bounded (independently of m), then we could pass to the limit and get $=0$.

So our plan is:

(1) approximate u, v by smooth functions u_m, v_m such that

$$\left.\begin{array}{l} u_m \to u \\ v_m \to v \\ F_m \to F \end{array}\right) \text{ locally in } L_1$$

(2) show that for $\forall\, \varphi \in C^1_0$ there exists $\psi^m \in C^1_0$ – a solution of $\psi^m_t + F_m \psi^m_x = \varphi$ and it's derivative ψ^m_x is bounded (independently of m)

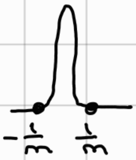
We will use entropy ineq. (E) HERE!

Step (1): One of the classical ideas to get a "smoother" function from any function u is to use convolution with "good kernel".

Consider $\omega(x)$ the standard "hat" function (bump)

$$\omega(x) = \begin{cases} e^{-\frac{1}{|x|^2-1}}, & |x| \leq 1 \\ 0, & |x| \geq 1 \end{cases}$$

$\omega_m = c_m\, m\, \omega(mx)$ is a "hat" on the interval $[-\frac{1}{m}, \frac{1}{m}]$



Properties:
1) $\omega_m \in C^\infty(\mathbb{R})$ (exercise)
2) $\omega_m \geq 0$, $\operatorname{supp}(\omega_m) = [-\frac{1}{m}, \frac{1}{m}]$
3) $\int_{\mathbb{R}} \omega_m = 1$
4) $\omega_m \xrightarrow[m\to\infty]{} \delta(x)$

Let $u_m = u * \omega_m$ and $v_m = v * \omega_m$, where

$$(f*g)(y) = \int_{\mathbb{R}} f(x)\, g(y-x)\, dx \quad \text{— convolution}$$

Have in mind such a picture

$f*g$ at point y is just averaging of f in a small neighbourhood of point y.

See 3blue 1brown about convolution

Properties of $u_m = u * \omega_m$:

First studied by Kurt Otto Friedrichs (1944) and Sergey Sobolev (1938)

ⓐ $u \in L^1_{loc} \Rightarrow u_m \in C^\infty$

ⓑ $u_m \to u$ in L^1_{loc}

ⓒ $F_m \to F$ in L^1_{loc}.

Proof:

▶ a)

$$u_m(y) = \int_{\mathbb{R}} u(x)\,\omega_m(y-x)\,dx$$

$$\frac{u_m(y+h)-u_m(y)}{h} = \int_{\mathbb{R}} u(x)\cdot\frac{\omega_m(y+h-x)-\omega_m(y-x)}{h}\,dx$$

↓ Lebesgue theorem

$$\int_{\mathbb{R}} u(x)\cdot\frac{\partial}{\partial y}\omega_m(y-x)\,dx \quad \text{etc}$$

and $\omega_m \in C^\infty$

b)

$$u_m - u = \int_{\mathbb{R}} \omega_m(y-x)\,[u(x)-u(y)]\,dx =$$

$$= \int_{\mathbb{R}} \omega_m(z)\,[u(y+z)-u(y)]\,dz =$$

$$= \int_{-\frac{1}{m}}^{1/m} \omega_m(z)\,[u(y+z)-u(y)]\,dz$$

$$\Rightarrow \int_K |u_m-u|\,dy \le \int_K dy \int_{-\frac{1}{m}}^{\frac{1}{m}} \omega_m(z)\,[u(y+z)-u(y)]\,dz$$

K - compact set

$$\le \underbrace{\int_{-\frac{1}{m}}^{1/m}\omega_m(z)\,dz}_{\substack{\shortparallel \\ 1}} \cdot \underbrace{\sup_{|z|<\frac{1}{m}} \int_K [u(y+z)-u(y)]\,dy}_{\substack{\downarrow\, m\to\infty \\ 0}}$$

$\Rightarrow$ $u_m \to u$ in L^1_{loc}.

c) Write F_m as follows:

$$F_m(x,t) = \frac{f(u_m)-f(v_m)}{u_m-v_m} = \frac{1}{u_m-v_m}\int_{u_m}^{v_m} f'(s)\,ds = \int_0^1 f'(u_m\theta + v_m(1-\theta))\,d\theta$$

Analogously, $F(x,t) = \int_0^1 f'(u\Theta + v(1-\Theta))\,d\Theta$.

Let $c := \max\limits_{|u|\le M} |f''(u)|$. Then

$$F - F_m = \int_0^1 \Big[f'(u\Theta + (1-\Theta)v) - f'(u_m\Theta + (1-\Theta)v_m)\Big]\,d\Theta =$$

$$= \int_0^1 f''(\xi)\,\big[\Theta(u-u_m) + (1-\Theta)(v-v_m)\big]\,d\Theta, \text{ where}$$

ξ is between $\Theta u + (1-\Theta)v$ and $\Theta u_m + (1-\Theta)v_m$.

Due to estimates $|u|, |v|, |u_m|, |v_m| \le M$, we have $|\xi| \le M$.

Thus,

$$|F(x,t) - F_m(x,t)| \le c \int_0^1 \big[\Theta\,|u-u_m| + (1-\Theta)\,|v-v_m|\big]\,d\Theta \le$$

$$\le c\,(|u-u_m| + |v-v_m|)$$

Then for any compact set K in $\{t \ge 0\}$

$$\iint_K |F(x,t) - F_m(x,t)| \le c \underset{\substack{\downarrow \\ 0}}{\iint_K |u-u_m|} + c\cdot \underset{\substack{\downarrow \\ 0}}{\iint_K |v-v_m|} \to 0$$

└ ∎

Lecture 11: Let's finish proving uniqueness of entropy sol.

Reminder: Scalar conservation law : $\begin{cases} u_t + (f(u))_x = 0 \\ u|_{t=0} = u_0(x) \end{cases}$ $(*)$

- $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ - bounded, measurable
  $f : \mathbb{R} \to \mathbb{R}$, $f \in C^2$, $f'' > 0$.

We understand solutions in weak sense :

$$\iint_{t>0} [u\varphi_t + f(u)\varphi_x]\, dx\, dt + \int_{t=0} u_0 \varphi\, dx = 0 \qquad (**)$$

for every test function $\varphi \in C_0^1$.

Thm 2 (!) :

Let u, v be 2 solutions of $(**)$, satisfying entropy condition $(E) : \exists E\ \forall a > 0,\ t > 0,\ x \in \mathbb{R}$

$$\frac{u(x+a) - u(x)}{a} < \frac{E}{t}. \qquad (E)$$

Then $u = v$ almost everywhere in $t > 0$.

Proof:

Γ▶ Our plan is as follows:

① We want to show that $\forall \varphi \in C_0^1$:

$$\iint_{t>0} (u-v)\varphi = 0 \quad [\Rightarrow \quad u = v \quad \text{a.e.}]$$

From $(**)$ we have $\iint_{t>0} (u-v)[\psi_t + F(x,t)\psi_x] = 0$
$\forall \psi \in C_0^1$

$$\text{for} \quad F(x,t) = \frac{f(u(x,t)) - f(v(x,t))}{u(x,t) - v(x,t)}.$$

So if $\forall \varphi \in C_0^1\ \exists \psi \in C_0^1$ such that

$$\psi_t + F(x,t)\psi_x = \varphi$$ – we would be done!

Unfortunately this is not true as u, v can be discontinuous and F is not necessarily smooth.

We need to use a PDE trick – "smoothen" u, v

② Consider

$$u_m = u * \omega_m \in C^\infty \quad ; \quad u_m \xrightarrow{L^1} u$$

$$v_m = v * \omega_m \in C^\infty \quad ; \quad v_m \xrightarrow{L^1} v$$

$$F_m = \frac{f(u_m) - f(v_m)}{u_m - v_m} \quad ; \quad F_m \xrightarrow{L^1} F$$

We have identity: fix $\varphi \in C_0^1$: it is enough to prove

$$\iint_{t \geq 0} (u-v)\,\varphi = \iint_{t\geq 0} (u-v)\,[F_m - F] \cdot \psi_x^m \ \boxed{\xrightarrow[m\to\infty]{} 0}$$

where ψ_m is the solution of the equation:

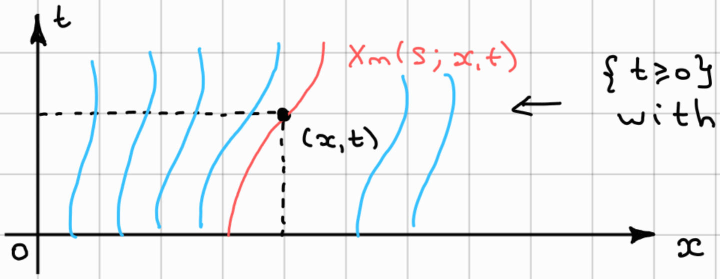
$$(M_1) \begin{cases} \psi_t^m + F_m(x,t)\,\psi_x^m = \varphi \\ \psi^m(x,T) = 0 \end{cases}$$

Here we may choose T so big such that $\varphi(x,t) = 0$ for $t \geq T$.

Notice that as F_m at least C^1, we obtain that the characteristic ODE :

$$\begin{cases} \dfrac{dx_m}{ds} = F_m(x_m, s) \\ x_m\big|_{s=t} = x \end{cases}$$

has a unique solution $x_m(s)$. It will be important for us the initial point (x,t), so we will denote such solution $x_m(s; x, t)$.



← $\{t \geq 0\}$ is folliated with characteristics

Lemma (solution to inhomogeneous transport equation)
The solution of the problem (M_1) is given by:

$$\psi^m(x,t) = \int_T^t \varphi(x_m(s;x,t), s)\,ds.$$

Proof: This is once again Duhamel principle!

Indeed, let's check directly:

$$\psi^m_t = \underbrace{\varphi(x_m(t;x,t),t)}_{\varphi(x,t)} + \int_T^t \frac{d}{dt}\varphi(x_m(s;x,t),s)\,ds$$

$$\psi^m_x = \int_T^t \frac{d}{dx}\varphi(x_m(s;x,t),s)\,ds$$

Thus, $$\psi^m_t + F\psi^m_x = \varphi + \underbrace{\int_T^t \left[\frac{d}{dt} + F\frac{d}{dx}\right]\varphi(x_m(s;x,t),s)\,ds}_{=0}$$

Indeed, $\left[\frac{d}{dt} + F\frac{d}{dx}\right]$ is the derivative along the characteristics. But if we move the starting point (x,t) along characteristics, the function φ does not change $\Rightarrow$ $\left[\frac{d}{dt} + F\frac{d}{dx}\right]\varphi(x_m(s;x,t),s) = 0 \quad \forall s.$

Corollary of Lemma: $\psi^m \in C^1_0(t \geq 0)$.

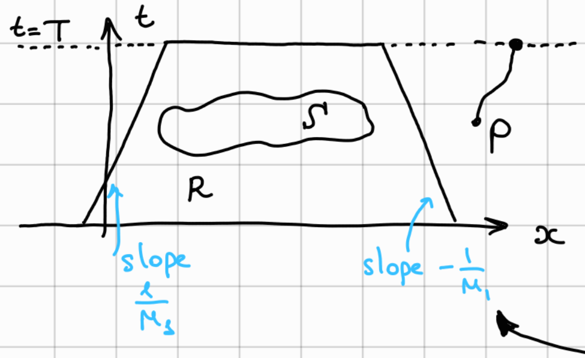
Proof:

By Lemma $\psi^m = \int_T^t \varphi(x_m(s;t,x),s)\,ds$

As $\varphi \in C^1(t\geq 0) \Rightarrow \psi^m \in C^1(t \geq 0)$.

Why ψ^m has a compact support?

Let S be support of φ (as $\varphi \subset C^1_0$)



As $F_m(x,t) = \int_0^1 f'(\theta u_m + (1-\theta)v_m)\,d\theta$

$\Rightarrow$ $|F_m| < M_1$ (as $f \in C^2$)

Consider a trapezoid R as on the figure:

(a) $S \subset R$

(b) R is bounded by four lines: $t=0$, $t=T$

and $t = -\frac{1}{M_1}x + const_1$; $t = \frac{1}{M_1}x + const_2$

Let's show that $\psi^m \equiv 0$ out of R:

1. $\psi^m = 0$ for $t \geq T$ because $\varphi \equiv 0$ there
2. Take $P = (x_1, t_1) \notin R$, $t_1 < T$.

$$x_m(s; x_1, t_1) \notin R \quad \forall s \quad \Rightarrow \quad x_m(T; x_1, t_1) \notin R$$

$$\Rightarrow \quad \varphi(x_m(s; x_1, t_1), s) = 0 \quad \forall s \quad \Rightarrow \quad \psi^m = 0. \quad \blacksquare$$

Lemma (boundedness of $|\psi^m_x|$)

$\exists\ C$ (independent of m):

$$|\psi^m_x| < C$$

Proof:

The main ingredient of proof is the entropy condition: $\forall a > 0,\ t > 0$

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}.$$

We see that close to $t = 0$ the entropy condition spoils ($\frac{E}{t} \to \infty$ as $t \to 0$).

Let $\alpha > 0$ be arbitrary. Then for $\forall\ t \geq \alpha$ the function $u(x,t) - \frac{Ex}{\alpha}$ is non-increasing in x:

$$u(x+a,t) - \frac{E(x+a)}{\alpha} - u(x,t) + \frac{Ex}{\alpha} \leq \frac{Ea}{t} - \frac{Ea}{\alpha} = Ea\underset{\leq 0}{\underbrace{\left(\frac{1}{t} - \frac{1}{\alpha}\right)}}$$

In what follows we will consider 2 cases:

ⓐ $t \geq \alpha$ ⓑ $0 \leq t \leq \alpha$

$\boxed{\text{Case } t \geq \alpha}$

Claim: $\frac{\partial u_m}{\partial x} \leq \frac{E}{\alpha}$; $\frac{\partial v_m}{\partial x} \leq \frac{E}{\alpha}$

and $\exists\ K = K_\alpha$: $\frac{\partial F_m}{\partial x} \leq K_\alpha$

Indeed, the function $\omega_n * \left(u - \frac{Ex}{\alpha}\right) = u_m - \frac{E\, \omega_n * x}{\alpha}$ is also non-increasing (and smooth)

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(u_m - \frac{E\,\omega_n * x}{\alpha}\right) = \frac{\partial u_m}{\partial x} - \frac{E}{\alpha} \leq 0$$

Analogously, $\frac{\partial v_m}{\partial x} \le \frac{E}{\alpha}$.

$$\frac{\partial F_m}{\partial x} = \int_0^1 f''(\theta u_m + (1-\theta) v_m)\left[\theta \frac{\partial u_m}{\partial x} + (1-\theta)\frac{\partial v_m}{\partial x}\right] d\theta$$

$$\Rightarrow \quad \frac{\partial F_m}{\partial x} \le \int_0^1 f''(\theta u_m + (1-\theta) v_m)\left[\theta \frac{E}{\alpha} + (1-\theta)\frac{E}{\alpha}\right] d\theta$$

$$= \frac{E}{\alpha}\int_0^1 f''(\theta u_m + (1-\theta) v_m)\, d\theta$$

There fore, $\frac{\partial F_m}{\partial x} \le K_\alpha = \frac{E}{\alpha}\max_{|u|\le M} f''(u)$. ■

$\llcorner$

Let's use this to prove $\left|\frac{\partial \varphi^m}{\partial x}\right| \le C, t \ge \alpha$

$$\frac{\partial \varphi^m}{\partial x} = \int_T^t \underbrace{\frac{\partial \phi}{\partial x_m}}_{\text{is bounded}} \cdot \frac{\partial x_m}{\partial x}(s; x, t)\, ds$$

Let's examine $\frac{\partial x_m}{\partial x}$.

For convinience, denote $a_m(s) = \frac{\partial x_m}{\partial x}(s; x, t)$

Here (x,t) - some fixed point in $\{t>0\}$.

Notice $x_m(t; x, t) = x$

$$\Rightarrow \quad a_m(t) = \frac{\partial x_m}{\partial x} = 1.$$

How $a_m(s)$ is changing with s?

$$\frac{\partial a_m}{\partial s} = \frac{\partial}{\partial s}\frac{\partial x_m}{\partial x} = \frac{\partial}{\partial x}\frac{\partial x_m}{\partial s} = \frac{\partial}{\partial x} F_m(x_m, s) =$$

$$= \frac{\partial}{\partial x} F_m(x_m(s; x, t), s) = \frac{\partial F_m}{\partial x} \cdot \frac{\partial x_m}{\partial x} =$$

$$= \frac{\partial F_m}{\partial x} \cdot a_m \quad \Rightarrow \quad \boxed{\frac{\partial a_m}{\partial s} = \frac{\partial F_m}{\partial x} \cdot a_m}$$

We can solve it: $a_m(s) = \exp\left(\int_t^s \frac{\partial F_m}{\partial x}(x_m(\tau), \tau)\, d\tau\right)$

Since we have $\alpha \le t \le s \le T$

$$\left|\frac{\partial x_m}{\partial x}\right| = |a_m(s)| = a_m(s) \le e^{K_\alpha(s-t)} \le e^{K_\alpha(T-\alpha)}$$

Thus, $\left| \frac{\partial \varphi^m}{\partial x} \right| \leq \int_T^t \left| \frac{\partial \Phi}{\partial x} \right| \cdot \left| \frac{\partial x_m}{\partial x} \right| ds \leq$

$$\leq (T-\alpha) \cdot C_1 \cdot e^{K_2 (T-\alpha)} =: C$$

The most important is that C does not depend on m!

$\boxed{\text{Case} \quad 0 \leq t \leq \alpha}$ Consider the total variation of φ^m as a function of x

$$V_t(\varphi^m) = \int_{\mathbb{R}} \left| \frac{\partial \varphi^m}{\partial x} \right| dx \qquad \text{for each fixed } t>0$$

As $\varphi^m \in C_0^1$ and for $t \geq \alpha$ $\left| \frac{\partial \varphi^m}{\partial x} \right| \leq C$ we have

$$V_t(\varphi^m) \leq C_\alpha \, , \quad t \geq \alpha$$

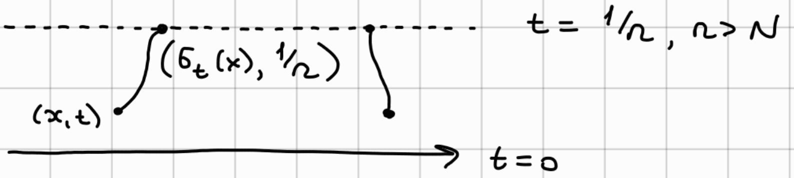
↳ does not depend on m.

Rmk: let's show that $\exists N \ \forall n > N$

$$V_t(\varphi^m) \leq C_{1/n} \quad \forall t : 0 < t < \frac{1}{n} < \frac{1}{N}$$

Since Φ has a compact support in $\{t>0\}$, there exists N : $\Phi(x,t) = 0$ if $t < 1/N$.

Thus, $\varphi^m_t + F_m \varphi^m_x = 0$ if $t < 1/N$



Let $\sigma_t : \mathbb{R} \to \mathbb{R}$ - bijection that takes φ^m at time t as initial condition and sends it to solution φ^m at time $t = \frac{1}{n}$. As φ^m is constant along characteristics, it is clear that

$$\sum_{k=1}^{p-1} |\varphi^m(x_{k+1}, t) - \varphi^m(x_k, t)| = \sum_{k=1}^{p-1} \left| \varphi^m\left(\sigma_t(x_{k+1}), \frac{1}{n}\right) - \varphi^m\left(\sigma_t(x_k), \frac{1}{n}\right) \right| \leq$$

for any finite sequence $x_1 < x_2 < \ldots < x_p$

$$\leq V_{1/n}(\varphi^m) \leq C_{1/n}.$$

Let's complete the proof of thm 2.

Fix $\varepsilon > 0$ - arbitrary. Take N from Rmk above. Choose $\alpha > 0$ so small s.t. $\alpha < \frac{1}{n} \leq \frac{1}{N}$ and

$$4 M M_1 C_{1/n} \, \alpha < \frac{\varepsilon}{2}.$$

For this α choose $\tilde{M}$ so large that

$$\iint_{t > \alpha} |u-v| \cdot |F_m - F| \cdot |\varphi_x^m| < \frac{\varepsilon}{2} \quad \text{if} \quad m \geq \tilde{M}$$

This can be done since $|u-v| \leq 2M$, $\left|\frac{\partial \varphi^m}{\partial x}\right| \leq K_\alpha$ and $F_m \to F$ in L^1_{loc}.

Then

$$\left| \iint_{t \geq 0} (u-v)\varphi \right| \leq \iint_{t \geq \alpha} + \iint_{t < \alpha} \underset{\substack{\uparrow \\ \text{see}}}{\leq} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

below

Now since $\alpha < \frac{1}{n} \leq \frac{1}{N}$

$$\iint_{t < \alpha} |u-v| \cdot |F_m - F| \cdot |\varphi_x^m| \leq 2M \cdot 2M_1 \iint_{t < \alpha} |\varphi_x^m| = 4MM_1 \int_0^\alpha \int_{\mathbb{R}} |\varphi_x^m|$$

$$= 4MM_1 \int_0^\alpha V_t(\varphi^m)\, dt \leq 4MM_1 \, C_{1/n} \, \alpha < \frac{\varepsilon}{2}$$

Thus, $\iint_{t \geq 0} (u-v)\varphi = 0 \quad \forall \ \varphi \in C_0^1 \implies u = v$ a.e. ■

Lecture 12. Riemann problem:

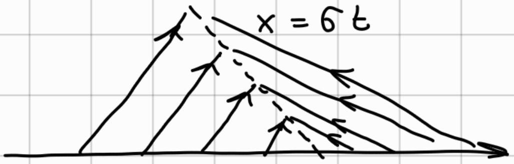
$$(RP)\quad \begin{cases} u_t + (f(u))_x = 0 \\ u(x,0) = \begin{cases} u_\ell, & x<0 \quad \text{- left state} \\ u_r, & x>0 \quad \text{- right state} \end{cases} \end{cases}$$

As before assume $f \in C^2$, $f'' > 0$.

Theorem (solution to a Riemann problem):

(i) If $u_\ell > u_r$, the unique entropy solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_\ell, & \text{if } x/t < \sigma \\ u_r, & \text{if } x/t > \sigma \end{cases}$$

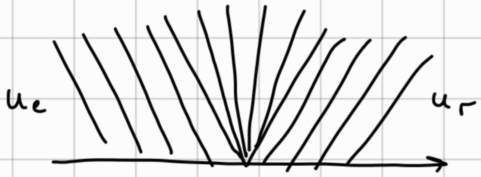


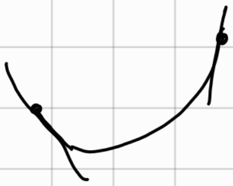
where $\sigma = \dfrac{f(u_\ell) - f(u_r)}{u_\ell - u_r}$

(ii) If $u_\ell < u_r$, the unique entropy solution is

$$u(x,t) = \begin{cases} u_\ell & \text{if } x/t < F'(u_\ell) \\ (F')^{-1}(x/t) & \text{if } F'(u_\ell) < x/t < F'(u_r) \\ u_r & \text{if } x/t > F'(u_r) \end{cases}$$

Such solution is called rarefaction wave





Proof:

▶ (i) As this is shock "down" it satisfies entropy condition $\Rightarrow$ this is a unique entropy sol.

(ii) Let's look for solution of the form:

$$u(x,t) = v\left(\frac{x}{t}\right) \Rightarrow u_t + (f(u))_x = -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(v)v'\frac{1}{t}$$
$$= v'\left(\frac{x}{t}\right)\frac{1}{t}\left(f'(v) - \frac{x}{t}\right)$$

If v' never vanishes $\Rightarrow f'(v) = \frac{x}{t} \Rightarrow v = (f')^{-1}\left(\frac{x}{t}\right)$

Also it is easy to check that v satisfies entropy cond. ■

Systems of conservation laws.

The most general: $u = \vec{u}(x,t) = (u_1(x,t), \dots, u_m(x,t))$

$$x \in \mathbb{R}^n, \; t \geq 0$$

$$\frac{d}{dt} \int_U u(x,t)\, dx = -\int_{\partial U} F(u)\nu\, dS = -\int_U \operatorname{div} F(u)\, dx$$

$$\Rightarrow \begin{cases} u_t + \operatorname{div} F(u) = 0, & x \in \mathbb{R}^n \\ u\big|_{t=0} = u_0 \end{cases} \qquad (*)$$

$u \in \mathbb{R}^m$ - state space. We will consider only

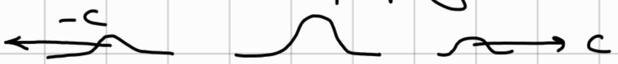
$F \in \mathbb{R}^m$ - flux $\qquad x \in \mathbb{R} \; (n=1)$

Example 1: (linear) wave equation: $u_{tt} - c^2 u_{xx} = 0$

$$U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + A U_x = 0 \qquad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}$$

Eigenvalues of A: $\lambda_\pm = \pm c$

correspond to propagation modes



Example 2: (non-linear) wave equation:

$$u_{tt} - (p(u_x))_x = 0$$

$$U = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \Rightarrow U_t + F(U)_x = 0$$

$$\begin{cases} u_{xt} - u_{tx} = 0 \\ u_{tt} - (p(u_x))_x = 0 \end{cases} \qquad F(U) = \begin{pmatrix} -u_t \\ -p(u_x) \end{pmatrix}$$

$$U = \begin{pmatrix} v \\ w \end{pmatrix} \Rightarrow \begin{cases} v_t - w_x = 0 \\ w_t - (p(v))_x = 0 \end{cases}$$

$$F(U) = \begin{pmatrix} -w \\ -p(v) \end{pmatrix}$$

This system is called p-system (or isentropic gas dynamics)

Example 3: Euler eqs for compressible gas flow:

$$\rho_t + (\rho v)_x = 0$$ (conservation of mass)

$$(\rho v)_t + (\rho v^2 + p)_x = 0$$ (conservation of momentum)

$$(\rho E)_t + (\rho E v + pv)_x = 0$$ (conservation of energy)

Unknowns:

- ρ – mass density
- v – velocity
- E – energy

$$p = (\rho, e)$$

$$E = e + \frac{v^2}{2}$$

internal energy

kinetic energy

$$U = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} \Rightarrow$$ can be written as $U_t + F(U)_x = 0$

Weak solutions:

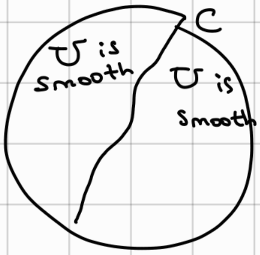
Let $v : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^m$ – smooth (C^1) with compact support, $v = (v^1, \dots, v^m)$

Do standard procedure: multiply the eq. by v and integrate by parts:

$$(**) \quad \int_0^\infty \int_{\mathbb{R}} [u \cdot v_t + F(u) v_x] \, dx \, dt + \int_{\mathbb{R}} u_0 \cdot v \, dx = 0$$

Def: we say $u \in L^\infty(\mathbb{R} \times (0, +\infty); \mathbb{R}^m)$ is a weak solution of (*) provided (**) holds for all v as above.

Lemma (Rankine-Hugoniot condition)



U has a jump discontinuity at C parametrized by smooth function $s(\cdot) : [0, +\infty) \to \mathbb{R}$ $(x,t) = (s(t), t)$ and let U_e be left values of U along the curve C;

U_r be right values of U along the curve C.

Then:

$$F(U_e) - F(U_r) = \dot{s}\,(U_e - U_r) \qquad (RH)$$

Rmk 1: proof is totally analogous to the scalar case - we omit it

Rmk 2: this equality (RH) is vector!

- What fluxes are reasonable?

Consider a wider class of semilinear systems

(SL) $\quad u_t + B(u)\, u_x = 0 \quad , \quad B : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$

If our solutions of $(*)$ are smooth, this system $(*)$ is equivalent to $U_t + DF \cdot U_x = 0$

$$B = DF = \begin{pmatrix} F^1_{z_1} & \cdots & F^1_{z_m} \\ \vdots & \ddots & \vdots \\ F^m_{z_1} & \cdots & F^m_{z_m} \end{pmatrix}_{m \times m}$$

Let's find formally the solutions in the form of a travelling wave:

$$u(x,t) = v(x - \sigma t) \Rightarrow -\sigma v' + B(v) v' = 0$$

Here v - profile, σ - velocity | Observe that this means that σ is eigenvalue of $B(u)$ and v' is an eigenvector.

If we want have some waves propagating, we should make some sort of "hyperbolicity" condition.

Def: If for each $z \in \mathbb{R}^m$ all eigenvalues of $B(z)$ are real and distinct, we call the system (SL) strictly hyperbolic.

From now on we will assume the system (SL) always strictly hyperbolic. We will write

(i) $\lambda_1(z) < \lambda_2(z) < \dots < \lambda_m(z)$, $z \in \mathbb{R}^m$ real and distinct eigenvalues of $B(z)$

(ii) $r_k(z)$ - eigenvectors of $B(z)$, $k=1\dots m$

$$B(z)\, r_k(z) = \lambda_k(z)\, r_k(z)$$

Strict hyperbolicity $\Rightarrow \mathrm{span}\{r_1(z), \dots, r_m(z)\} \equiv \mathbb{R}^m$ $\forall z \in \mathbb{R}^m$

(iii) $\ell_k(z)$ - eigenvectors of $B^T(z)$, correspond. to $\lambda_k(z)$

$$B^T(z)\, \ell_k(z) = \lambda_k(z)\, \ell_k(z)$$

or

$$\ell_k\, B(z) = \lambda_k\, \ell_k$$

Thus, we can regard r_k as right eigenvectors
ℓ_k as left eigenvectors

Rmk: $r_k \cdot \ell_s = 0$ if $k \neq \ell$

Indeed,

$$\lambda_k\,(\ell_s \cdot r_k) = \ell_s \cdot (\lambda_k r_k) = \ell_s\,(B \cdot r_k) = (\ell_s B)\, r_k =$$
$$= (\lambda_s \ell_s)\, r_k = \lambda_s \cdot \ell_s r_k$$

As $\lambda_k \neq \lambda_s \Rightarrow \ell_s \cdot r_k = 0$, $k \neq s$

Let us formulate some theorems that sound reasonable (without proof):

Theorem (invariance of hyperbolicity under change of coordinates)

Let u be smooth solution of (SL)
Assume $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be a smooth diffeo
Ψ it's inverse

Then: $\tilde{u} = \Phi(u)$ solves the strictly hyperbolic system: $\tilde{u}_t + \tilde{B}(\tilde{u})\, \tilde{u}_x = 0$

for $\tilde{B}(\tilde{z}) = D\Phi(\Psi(\tilde{z}))\, B(\psi(\tilde{z}))\, D\psi(\tilde{z})$

Rmk: weak solutions are not preserved under smooth nonlinear transformations of the equations: consider scalar eq: $u_t + (f(u))_x = 0$

$f'' > 0$ $\qquad u \mapsto v = f'(u)$

$$\begin{aligned} v_t &= f''(u)\cdot u_t \\ v_x &= f''(u)\cdot u_x \end{aligned} \quad \Rightarrow \quad v_t + v\cdot v_x = 0$$

Burgers!

But this map doesn't map discontinuous solutions into themselves. Just write RH condition: the original eq: $s = \dfrac{f(u_r) - f(u_l)}{u_r - u_l}$

and for the transformed eq: $s = \dfrac{f'(u_l) - f'(u_r)}{u_l - u_r}$

Theorem (dependence of eigenvalues and eigenvectors on parameters)

Assume matrix function B is smooth, strictly hyperbolic. Then:

(i) the eigenvalues $\lambda_k(z)$ depend smoothly on z

(ii) we can select the right eigenvectors $r_k(z)$ and left eigenvectors $\ell_k(z)$ to depend smoothly on $z \in \mathbb{R}^m$ and satisfy the normalization:

$$|r_k(z)| = 1, \quad |\ell_k(z)| = 1$$

Example 1 (continued): $c \neq 0 \Rightarrow$ system is strictly hyperbolic

Example 2 (continued): $p' > 0 \Rightarrow$ system is strictly hyperbolic

$$D\begin{pmatrix} -w \\ -p(v) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -p' & 0 \end{pmatrix} \qquad \lambda_{\pm} = \pm\sqrt{p'}$$

Riemann problem (RP) :

$$\begin{cases} u_t + \big(F(u)\big)_x = 0, & u \in \mathbb{R}^m \\ u(x,0) = \begin{cases} u_l, & x<0 \\ u_r, & x>0 \end{cases} \end{cases}$$

We will call u_l, u_r left and right initial states

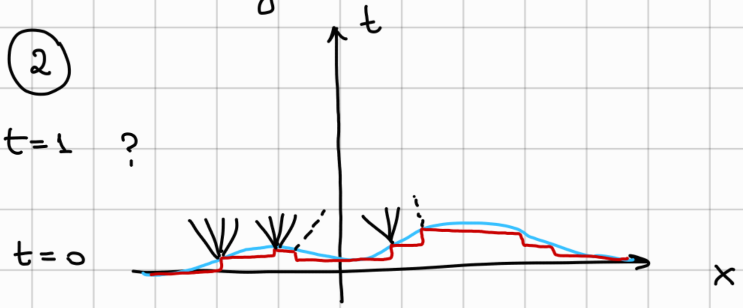
We aim at finding exact solutions to a Riemann problem. Why they are useful ?

①

Often the solution to a Riemann problem appear as limiting one when $t \to +\infty$.

$u' = f(u)$ - steady states : $f(u)=0$ $\qquad$ $\varepsilon u_{xx} = u_t + (f(u))_x$ - steady states: $u_t + (f(u))_x = 0$

②



One can approximate initial condition by piecewise constant initial data and solve many Riemann problems.

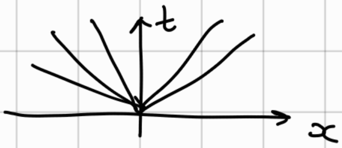
The obtained solution is some approximation

So using RP one can prove existence of solutions to Cauchy problem (with arbitrary initial data)

Rmk : notice that both equation and initial condition in RP stay the same if we consider $(x,t) \longmapsto (\alpha x, \alpha t)$

Thus the solution depends only on $\frac{x}{t}$

it is constant on rays $t = kx$



Let us be engineers: to construct the general solution we need "building blocks":

- → smooth solutions ⇝ rarefaction waves
- → discontinuous solutions ⇝ shock waves
- → constant states.

§ Simple waves:

$$u(x,t) = v(w(x,t))$$

$$\left.\begin{array}{l} v: \mathbb{R} \to \mathbb{R}^m \\ w: \mathbb{R}\times(0,+\infty) \to \mathbb{R} \end{array}\right\} \text{to be found}$$

$$u_t + F(u)_x = 0 \quad \Rightarrow \quad \dot v \cdot w_t + \underbrace{DF\cdot \dot v}_{\overset{?}{=}\lambda_k \dot v}\cdot w_x = 0$$

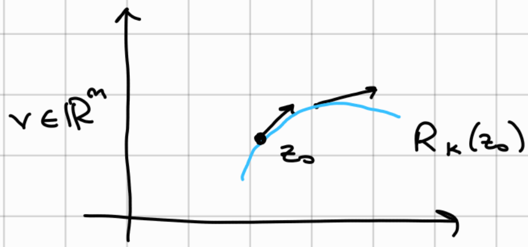
$$\Uparrow$$

$$(SW) \begin{cases} w_t + \lambda_k(v(w))\, w_x = 0 \\ \dot v = r_k(v(s)) \end{cases}$$

Def: $u(x,t) = v(w(x,t))$ is called a simple wave if (SW) holds.

The main point is that we can consider first $\dot v = r_k(v)$, and then regard $w_t + \lambda_k w_x = 0$ as a scalar conservation law!

Def: given a fixed state $z_0 \in \mathbb{R}^m$, we define k^{th}-rarefaction curve $R_k(z_0)$ to be path in $\mathbb{R}^m$ of the solution of the ODE $\dot v = r_k(v)$ which passes through point z_0.



Given solution R_k we can rewrite PDE as

$$w_t + F_k(w)_x = 0$$

for $F_k(s) = \int_0^s \lambda_k(v(t))\, dt$

If F_k is convex, we know that the solution exists and is unique.
So this PDE will fall into general theory provided F_k is strictly convex. Let us therefore compute:

$$F_k'(s) = \lambda_k(v)\cdot \dot{v} = \lambda_k(v)$$

$$F_k'' = D\lambda_k \cdot \dot{v} = \underbrace{D\lambda_k(v(s))\cdot r_k(v(s))}_{\text{this is the derivative of } \lambda_k \text{ along the k-rarefaction curve}}$$

So F_k will be convex if

$$D\lambda_k(z)\cdot r_k(z) > 0 \qquad \forall z \in \mathbb{R}^m$$

F_k – concave if

$$D\lambda_k(z)\cdot r_k(z) < 0 \qquad \forall z \in \mathbb{R}^m$$

F_k – linear if

$$D\lambda_k(z)\cdot r_k(z) \equiv 0 \qquad \forall z \in \mathbb{R}^m$$

<u>Def</u>: (i) the pair $(\lambda_k(z), r_k(z))$ is called genuinely nonlinear provided

$$D\lambda_k(z)\cdot r_k(z) \neq 0 \qquad \forall z \in \mathbb{R}^m$$

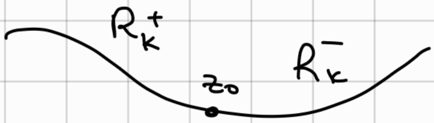
(ii) the pair $(\lambda_k(z), r_k(z))$ is called linearly degenerate provided

$$D\lambda_k(z)\cdot r_k(z) \equiv 0 \qquad \forall z \in \mathbb{R}^m$$

<u>Notation</u>: if the pair is genuinely nonlinear, write

$$R_k^+(z_0) = \{z \in R_k(z_0) : \lambda_k(z) > \lambda_k(z_0)\}$$

$$R_k^-(z_0) = \{z \in R_k(z_0) : \lambda_k(z) < \lambda_k(z_0)\}$$



$$R_k(z_0) = R_k^+(z_0) \cup \{z_0\} \cup R_k^-(z_0)$$

Lecture 13 : Reminder : we consider systems of conservation laws

$x \in \mathbb{R}$, $t > 0$, $U(x,t) = (u_1(x,t), \dots, u_m(x,t))$

$$(*) \quad U_t + F(U)_x = 0$$

- $U \in \mathbb{R}^m$ - state
- $F : \mathbb{R}^m \to \mathbb{R}^m$ - flux
- $DF \in M^{m \times m}$

Def: the system $(*)$ is called hyperbolic if $DF(U)$ has m real eigenvalues:

$$\lambda_1(U) \le \dots \le \lambda_m(U)$$

and the corresponding eigenvectors $r_i(U)$, $i = 1, \dots, m$, are linearly independent (form basis)

Def: the system $(*)$ is called strictly hyperbolic if $(*)$ is hyperbolic and all eigenvalues are distinct: $\lambda_1(U) < \dots < \lambda_m(U)$

In what follows we consider strictly hyperbolic systems of conservation laws.

- $U_t + B(U) U_x = 0$; eigenvalues of $B(U)$: $\lambda_1(U) < \dots < \lambda_m(U)$

$$B(U)\, r_i(U) = \lambda_i(U)\, r_i(U) \quad , i = 1 \dots m$$
$$l_i(U)\, B(U) = \lambda_i(U)\, l_i(U) \quad , i = 1 \dots m$$

Our goal for today: give a "constructive" proof that a Riemann problem

$$(RP) \quad U(x,0) = \begin{cases} U_l, & x < 0 \\ U_r, & x > 0 \end{cases}$$

has a solution if u_l and u_r are close.

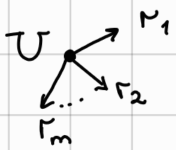
(local solution to a Riemann problem)

Our "building blocks": $i = 1 \dots m$

- i - rarefaction wave
- i -shock wave
- (i - contact discontinuity)
- constant states

Global picture:

$\mathbb{R}^m$: at each point U m eigenvectors



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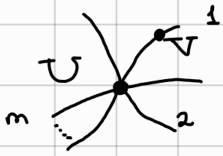
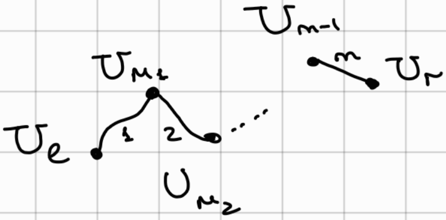


we will construct m C^1-smooth curves such that taking point V on i-curve would mean that we have eigher smooth or discontinuous solution from U to V "corresponding to i-characteristic" $\longleftrightarrow \lambda_i$

$\rightsquigarrow$



We can construct a sequence of waves:

$$U_e \underset{\substack{\uparrow \\ \text{1-charact.}}}{\longrightarrow} U_{M_1} \underset{\substack{\uparrow \\ \text{2-charac.}}}{\longrightarrow} U_{M_2} \to \dots \underset{\substack{\uparrow \\ \text{m-charac.}}}{\longrightarrow} U_r$$

§ Simple waves:

$$U(x,t) = V(w(x,t))$$

$w : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ - scalar

$V : \mathbb{R} \to \mathbb{R}^m$

$$\dot V w_t + \underbrace{DF(V(w)) \cdot \dot V}_{\lambda_k \cdot V} w_x = 0, \quad k = 1 \dots m$$

$$\begin{cases} \dot V = r_k(V(s)) \\ w_t + \underline{\lambda_k(V(w))} w_x = 0 \end{cases}$$

- scalar conservation law across the integral curve of the vector field induced by r_k

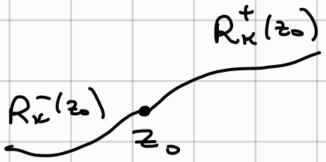
plays a role of the speed of propagation

$$\Updownarrow$$

$$w_t + (F_k(w))_x = 0 \quad \text{for} \quad F_k(s) = \int_0^s \lambda_k(V(s))\,ds$$

Def: the pair (λ_k, r_k) [or sometimes called k-characteristic family] is called genuinely nonlinear if $D\lambda_k(z) \cdot r_k(z) \neq 0 \quad \forall z \in \mathbb{R}^m$

- is called linearly degenerate if $D\lambda_k \cdot r_k \equiv 0$



$$R_k^+(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) > \lambda_k(z_0) \}$$
$$R_k^-(z_0) = \{ z \in R_k(z_0) : \lambda_k(z) < \lambda_k(z_0) \}$$
$$R_k(z) = R_k^+(z_0) \cup \{z_0\} \cup R_k^-(z_0)$$

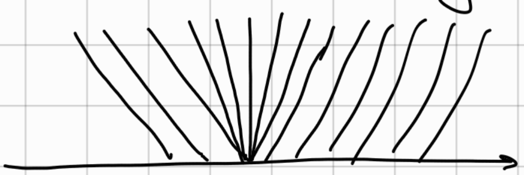
Thm (existence of k-rarefaction waves):

Suppose that for some $k = 1, \dots, m$:

(i) the pair (λ_k, r_k) is genuinely nonlinear

and (ii) $U_r \in R_k^+(z_0)$.

Then there exists a continuous integral solution U of a Riemann problem (RP), which is a k-simple wave constant along lines through origin



Rmk: if $U_r \in R_k^-$, then such a cont. sol. doesn't exist!

Proof:

1. Take $w_l, w_r \in \mathbb{R}$: $U_l = V(w_l)$; $U_r = V(w_r)$
   Suppose $w_l < w_r$.
2. Consider a scalar Riemann problem consisting of PDE
$$\begin{cases} w_t + (F_k(w))_x = 0 \\ w(x,0) = \begin{cases} w_l, & x<0 \\ w_r, & x>0 \end{cases} \end{cases}$$

$$F_k' = \lambda_k(V(s)), \quad F_k'' = D\lambda_k(V(s)) \cdot r_k(V(s)) \underset{(i)}{\neq} 0$$

(ii) $\Rightarrow$ $\lambda_k(U_r) > \lambda_k(U_l)$

$\Rightarrow$ $F_k'(w_r) > F_k'(w_l)$ $\Rightarrow$ F - strictly convex

$\Rightarrow$ this scalar conservation law admits a continuous solution – a rarefaction wave

$$w(x,t) = \begin{cases} w_l & \text{if } \frac{x}{t} < F_k'(w_l) \\ (F_k')^{-1}\left(\frac{x}{t}\right) & \text{if } F_k'(w_l) < \frac{x}{t} < F_k'(w_r) \\ w_r & \text{if } F_k'(w_r) < \frac{x}{t} \end{cases}$$

Thus $U(x,t) = V(w(x,t))$ solves PDE. The case $w_l > w_r$ is treated similarly (F_k is concave) ■

Shock waves : by RH condition , $\sigma \in \mathbb{R}$ - a shock wave speed

$$F(U_\ell) - F(U_r) = \sigma(U_\ell - U_r)$$

Def : for a given (fixed) state $U_0 \in \mathbb{R}^m$ we define a shock set (Hugoniot locus)

$$S(U_0) = \{ U \in \mathbb{R}^m : \exists \underset{\sigma = \sigma(U, U_0)}{\underset{\uparrow}{\sigma}} \in \mathbb{R} : F(U) - F(U_\ell) = \sigma(U - U_\ell) \}$$

That is this is a set of all states to which there possibly exist a shock wave (with some speed) from U_0.

Thm (structure of shock set)

Fix $U_0 \in \mathbb{R}^m$. In some neighborhood of U_0 $S(U_0)$ consists of the union of m smooth curves $S_k(U_0)$, $k = 1 \dots m$, with the following properties:

(i) The curve $S_k(U_0)$ passes through U_0 with tangent $r_k(U_0)$

(ii) $\lim\limits_{U \to U_0} \sigma(U, U_0) = \lambda_k(z_0)$

(iii) $\sigma(U, U_0) = \dfrac{\lambda_k(U) + \lambda_k(U_0)}{2} + O(|U - U_0|^2)$

as $U \to U_0$ with $U \in S_k(U_0)$.

Proof :

▶ $F(U) - F(U_0) = B(U)(U - U_0)$, where

$$B(U) = \int_0^1 DF(U_0(1-t) + Ut)\,dt \quad , U \in \mathbb{R}^m$$

— "averaged" Jacobi matrix DF

$U \in S(U_0)$ iff $\quad (B(U) - \sigma I)(U - U_0) = 0 \qquad (1)$

for some scalar $\sigma = \sigma(U, U_0)$.

$$B(U_0) = DF(U_0)$$

Strict hyperbolicity $\Rightarrow$ $\det(\lambda I - B(U_0))$ has m distinct real roots

$\Rightarrow$ $\det(\lambda I - B(U))$ has m distinct real roots if U is close to U_0.

Moreover, $\hat{\lambda}_1(U) < \dots < \hat{\lambda}_m(U)$ are smooth functions of U and $\hat{r}_k(U)$, $\hat{\ell}_k(U)$ unit vectors:

$$\hat{\lambda}_k(U_0) = \lambda_k(U_0)$$
$$\hat{r}_k(U_0) = r_k(U_0)$$
$$\hat{\ell}_k(U_0) = \ell_k(U_0)$$

and

$$B(U)\,\hat{r}_k(U) = \hat{\lambda}_k(U)\,\hat{r}_k(U) \qquad k = 1 .. m$$
$$\hat{\ell}_k(U)\,B(U) = \hat{\lambda}_k(U)\,\hat{\ell}_k(U)$$

Note that both $\{\hat{r}_k\}$ and $\{\hat{\ell}_k\}$ are bases of $\mathbb{R}^m$ and $\hat{r}_k \cdot \hat{\ell}_n = 0$, $n \neq k$.

Eq. (1) will hold provided $\sigma = \hat{\lambda}_k$ for some k and $U - U_0$ is parallel to $\hat{r}_k$. This is equivalent to:

$$\hat{\ell}_\ell(U) \cdot (U - U_0) = 0, \quad \ell \neq k$$

These are $(m-1)$ equations to m components of U, so we can use Implicit Function Theorem to solve it.

Define $\Phi_k : \mathbb{R}^m \to \mathbb{R}^{m-1}$

$$\Phi_k(U) = \left(\dots, \hat{\ell}_{k-1}(U)(U - U_0), \hat{\ell}_{k+1}(U)(U - U_0), \dots \right)$$

$\Phi_k(U_0) = 0$ and $D\Phi_k(U_0) = \begin{pmatrix} \ell_1(U_0) \\ \vdots \\ \ell_{k-1}(U_0) \\ \ell_{k+1}(U_0) \\ \dots \end{pmatrix}$

Since $\{l_i\}$ form a basis, we have

$$\operatorname{rank} D\Phi_K(U_0) = m-1$$

Hence, $\exists$ a smooth curve $\varphi_K : \mathbb{R} \to \mathbb{R}^m$ such that $\varphi_K(0) = U_0$ and

$$\Phi_K(\varphi_K(t)) = 0 \quad \forall\ t \text{ close to } 0.$$

The path of curve φ_K define $S_K(U_0)$
We may choose parametrization:

$$|\dot{\varphi}_K(t)| = 1$$

Thus we have found m smooth curves $S_K(U_0)$. Let us now properties (i)-(iii)

Property (i):

$$\varphi_K(t) - U_0 = \mu(t) \cdot \hat{r}_K(\varphi_K(t))$$

where μ is a smooth function satisfying $\mu(0) = 0$, $\dot{\mu}(0) = 1$

Thus, $\dot{\varphi}_K(0) = \hat{r}_K(U_0) = r_K(U_0)$ at U_0
Hence, the curve $S_K(U_0)$ has tangent $r_K(U_0)$

Property (ii): According to what we have proved, there exists a smooth function $\sigma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$: $\forall\ t$ close to 0

$$F(\varphi_K(t)) - F(U_0) = \sigma(\varphi_K(t), U_0)\,(\varphi_K(t) - U_0)$$

Thus,

$$DF(U_0) \cdot \dot{\varphi}_K(0) = \sigma(U_0, U_0)\,\dot{\varphi}_K(0)$$

$$\Rightarrow \quad \sigma(U_0, U_0) = \lambda_K(U_0)$$

Property (iii): for simplicity write $\sigma(t) = \sigma(\varphi_k(t), U_0)$

$$F(\varphi_k(t)) - F(U_0) = \sigma(t)\,(\varphi_k(t) - U_0).$$

Differenciate twice wrt t :

$$\frac{d}{dt} : \quad DF(\varphi_k(t)) \cdot \dot{\varphi}_k(t) = \dot{\sigma}\,(\varphi_k(t) - U_0) + \sigma \cdot \dot{\varphi}_k$$

$$\frac{d^2}{dt^2} : \quad \left(D^2F(\varphi_k(t)) \cdot \dot{\varphi}_k\right)\dot{\varphi}_k + DF(\varphi_k(t)) \cdot \ddot{\varphi}_k =$$

$$= \ddot{\sigma}\,(\varphi_k - U_0) + 2\,\dot{\sigma} \cdot \dot{\varphi}_k + \sigma\,\ddot{\varphi}_k$$

Evaluate this expression at $t=0$ $\left(\begin{array}{l} \varphi_k(0) = U_0 \\ \dot{\varphi}_k(0) = r_k(U_0) \end{array}\right)$

$$(2) \quad \left(D^2F(U_0)\, r_k(U_0) - 2\dot{\sigma}\, I\right) r_k(U_0) = \left(\lambda_k(U_0) - DF(U_0)\right) \cdot \ddot{\varphi}_k$$

Let $\psi_k(t) = V(t)$ be a unit speed parametrization of the rarefaction curve $R_k(U_0)$ near U_0.

Then $\psi_k(0) = U_0$, $\dot{\psi}_k(t) = r_k(\psi_k(t))$

Thus, $DF(\psi_k(t))\, r_k(t) = \lambda_k(t)\, r_k(t)$

Differenciate this wrt t and evaluate at $t=0$

$$(3) \quad \left(D^2F(U_0)\, r_k(U_0) - \dot{\lambda}_k(0)\, I\right) r_k(U_0) = -(DF + \lambda_k I)\,\dot{r}_k$$

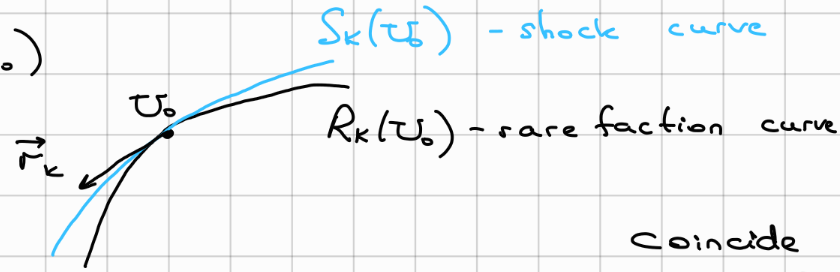
Subtract (3) from (2) and obtain:

$$\left(\dot{\lambda}_k(0) - 2\dot{\sigma}\right) r_k(U_0) = \left(DF - \lambda_k I\right)\left(\dot{r}_k - \ddot{\varphi}_k\right)$$

Take dot product with $\ell_k(U_0)$, we obtain

$$\dot{\lambda}_k(0) = 2\,\dot{\sigma}(0) \Rightarrow 2\,\sigma(t) = \lambda_k(U_0) + \lambda_k(U) + O(t^2) \blacksquare$$

So we have $S_k(U_0)$ and $R_k(U_0)$ agree at least to first order at U_0.



In the linearly degenerate case these curves coincide

Thm (linear degeneracy).

Suppose for some $k = 1 \dots m$ the pair (λ_k, r_k) is linearly degenerate. Then for each $U_0 \in \mathbb{R}^m$:

(i) $R_k(U_0) = S_k(U_0)$

(ii) $\sigma(U, U_0) = \lambda_k(U) = \lambda_k(U_0) \quad \forall U \in S_k(U_0)$

Proof:

⌈ Let $V = V(s)$ solve ODE

$$\begin{cases} \dot{V}(s) = r_k(V(s)) \\ V(0) = U_0 \end{cases}$$

Then as $D\lambda_k \cdot r_k \equiv 0$, the mapping $s \mapsto \lambda_k(V(s))$ is constant.

So

$$F(V(s)) - F(U_0) = \int_0^s DF(V(t)) \cdot \dot{V}(t)\,dt =$$

$$= \int_0^s DF(V(t)) \cdot r_k(V(t))\,dt = \int_0^s \lambda_k(V(t))\, r_k(V(t))\,dt$$

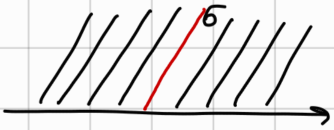
$$= \lambda_k(U_0) \int_0^s \dot{V}(t)\,dt = \lambda_k(U_0)(V(t) - U_0)$$

⌊ ∎

Contact discontinuities: Let (λ_k, r_k) be linearly degenerate

$U_l \in \mathbb{R}^m$, $U_r \in S_k(U_l)$

Then $U(x,t) = \begin{cases} U_l, & x < \sigma t \\ U_r, & x > \sigma t \end{cases}$ $\quad \sigma = \sigma(U_l, U_r) = \lambda_k(U_l)$



k- contact discontinuity

Shock waves: Let (λ_k, r_k) be genuinely nonlinear
$U_e \in \mathbb{R}^m$, $U_r \in S_k(U_e)$

Consider $U(x,t) = \begin{cases} U_e, & x<\sigma t \\ U_r, & x>\sigma t \end{cases}$ for $\sigma = \sigma(U_e, U_r)$

There are 2 essentialy different cases:
case I: $\lambda_k(U_r) < \lambda_k(U_e)$
case II: $\lambda_k(U_e) < \lambda_k(U_r)$

In view of thm of structure of shock curve, we have: case I: $\lambda_k(U_r) < \sigma(U_e, U_r) < \lambda_k(U_e)$
$\lambda_k(U_e) < \sigma(U_e, U_r) < \lambda_k(U_r)$
provided that U_r is close to U_e

Def: assume the pair (λ_k, r_k) is genuinely nonlinear at U_e. We say that the pair (U_e, U_r) is admissible provided:

ⓐ $U_r \in S_k(U_e)$

ⓑ $\lambda_k(U_r) < \sigma(U_e, U_r) < \lambda_k(U_e)$

We refer to this condition as Lax entropy condition.

Def: If (U_e, U_r) is admissible, we call our solution U defined as above a k-shock wave.

Def: Let $S_k^+(U_0) = \{ U \in S_k(U_0) : \lambda_k(U_0) < \sigma(U, U_0) < \lambda_k(U) \}$

$S_k^-(U_0) = \{ U \in S_k(U_0) : \lambda_k(U_0) > \sigma(U, U_0) > \lambda_k(U) \}$

Then $S_k(U_0) = S_k^+(U_0) \cup \{z_0\} \cup S_k^-(U_0)$

Note that the pair (U_e, U_r) is adm. iff $U_r \in S_k^-(U_e)$

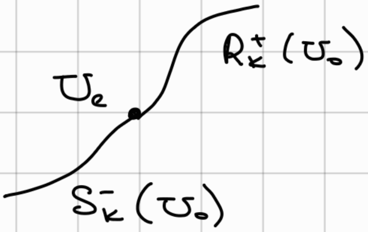
Now let us glue everything together.

Def: (i) if pair (λ_k, r_k) is genuinely nonlinear, write $T_k(U_0) = R_k^+(U_0) \cup \{U_0\} \cup S_k^-(U_0)$

(ii) if pair (λ_k, r_k) is linearly degenerate, write $T_k(U_0) = R_k(U_0) = S_k(U_0)$

Rmk: the curve $T_k(U_0)$ is C^1

So if $U_r \in T_k(U_\ell)$, then there exists a solution to a Riemann problem (being or k-rarefaction wave or k-shock wave or k-contact discontinuity)



Finally, we want to prove theorem:

Thm (local solution of Riemann problem)

Assume that for each $k = 1 \dots m$ the pair (λ_k, r_k) is either genuinely nonlinear or linearly degenerate. Suppose we have fixed U_ℓ. Then for each right state U_r sufficiently close to U_ℓ there exists an integral solution U of (RP) which is constant on lines through the origin.

Proof:

Again Implicit Function Theorem
(Next time)

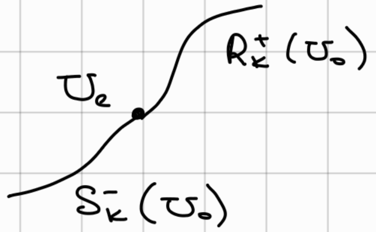
Now let us glue everything together.

Def: (i) if pair (λ_k, r_k) is genuinely nonlinear, write $T_k(U_0) = R_k^+(U_0) \cup \{U_0\} \cup S_k^-(U_0)$

(ii) if pair (λ_k, r_k) is linearly degenerate, write $T_k(U_0) = R_k(U_0) = S_k(U_0)$

Rmk: the curve $T_k(U_0)$ is C^1

So if $U_r \in T_k(U_l)$, then there exists a solution to a Riemann problem (being or k-rarefaction wave or k-shock wave or k-contact discontinuity)



Finally, we want to prove theorem:

Thm (local solution of Riemann problem)

Assume that for each $k = 1 \ldots m$ the pair (λ_k, r_k) is either genuinely nonlinear or linearly degenerate. Suppose we have fixed U_e. Then for each right state U_r sufficiently close to U_e there exists an integral solution U of (RP) which is constant on lines through the origin.

Proof:

▶ Again Implicit Function Theorem: $\Phi : \mathbb{R}^m \to \mathbb{R}^m$

First, for each family of curves T_k, $k=1..m$, choose the nonsingular parameter τ_k to measure arc length: $\forall\ U, \tilde{U} \in \mathbb{R}^m$ with

$\tilde{U} \in T_k(U)$ we have

$\tau_k(\tilde{U}) - \tau_k(U) =$ (signed) distance from $\tilde{U}$ to U along the curve $T_k(z)$

We take "$+$" if $\tilde{U} \in R_k^+(U)$ and
"$-$" if $\tilde{U} \in S_k^-(U)$.

Second, given $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ with $|t|$ small we define $\Phi(t) = U$ as follows.

- $\Phi(0) = U_0$
- Then choose states $U_1, \dots, U_m$:

$U_1 \in T_1(U_0)$, $\tau_1(U_1) - \tau_1(U_0) = t_1$
$U_2 \in T_2(U_1)$, $\tau_2(U_2) - \tau_2(U_1) = t_2$
....
$U_m \in T_m(U_{m-1})$, $\tau_m(U_m) - \tau_m(U_{m-1}) = t_m$

Now write $\Phi(t) = z_m$.

- $\Phi \in C^1$
- $\Phi(0) = z_0$
- $D\Phi(0)$ is nonsingular

$$\Phi(0, \dots, t_k, \dots, 0) - \Phi(0, \dots, 0) = t_k r_k(U_0) + o(t_k), t_k \to 0$$

Thus, $\frac{\partial \Phi}{\partial t_k}(0) = r_k(U_0)$ and so

$$D\Phi(0) = (r_1(U_0), \dots, r_m(U_0))_{m \times m}$$

Since $\{r_i\}$ is a basis, $D\Phi(0)$ is nonsingular.
Thus, by the inverse function theorem
$\forall U_r$ sufficiently close to U_l $\exists! \; t = (t_1, \dots t_m)$
$\Phi(t) = U_r$.

So we get a sequence: $U_l \to U_1 \to U_2 \to \dots \to U_m$.

Recall that if U_{k-1} and U_k are joined by k-rarefaction wave, this wave is:

$$\begin{cases} U_{k-1} & \text{if } \frac{x}{t} < \lambda_k(U_{k-1}) \\ (f_k')^{-1}(\frac{x}{t}) & \text{if } \lambda_k(U_{k-1}) < \frac{x}{t} < \lambda_k(U_k) \\ U_k & \text{if } \lambda_k(U_k) < \frac{x}{t} \end{cases}$$

Moreover, if U_{k-1}, U_k are joined by k-shock, it has the form:

$$\begin{cases} U_{k-1} & \text{if } \frac{x}{t} < \sigma(U_k, U_{k-1}) \\ U_k & \text{if } \frac{x}{t} > \sigma(U_k, U_{k-1}) \end{cases}$$

In both cases the waves are constant outside the regions $\lambda_k(U_0) - \varepsilon < \frac{x}{t} < \lambda_k(U_0) + \varepsilon$ for small $\varepsilon > 0$ provided U_k, U_{k-1} are close enough.

This is true for $k = 1, \dots, m$.

Since $\lambda_1(U_0) < \dots < \lambda_m(U_0)$, we see that rarefactions, shock or contact discontinuities connecting

U_0 to U_1 $\rightsquigarrow \approx \lambda_1(U_0)$

U_1 to U_2 $\rightsquigarrow \approx \lambda_2(U_0)$

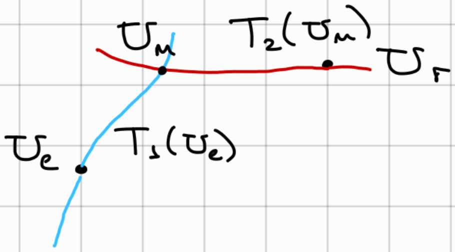
$\dots\dots$ $\dots$

U_{m-1} to U_m $\rightsquigarrow \approx \lambda_m(U_0)$

do not intersect.

$\llcorner$ Thus, we have constructed a solution. ■

$\boxed{m=2}$



Lecture 14: Solution to exercise 3 from HW 3.

$S_t + f(S)_x = 0$ $\quad$ f(S) – S-shaped

Buckley-Leverett eq

flow in porous media

- $f(0) = 0$, $f(1) = 1$
- $f' > 0$
- $\exists\, s = \frac{1}{2}$: $f''(s) > 0$, $s < \frac{1}{2}$

$f''(\frac{1}{2}) = 0$

$f''(s) > 0$, $s > \frac{1}{2}$.

$$S_t + \underbrace{f'(S)}_{\lambda(S)} S_x = 0$$

$D\lambda = f''(S)$ is 0 at point $s = \frac{1}{2}$

$\Downarrow$

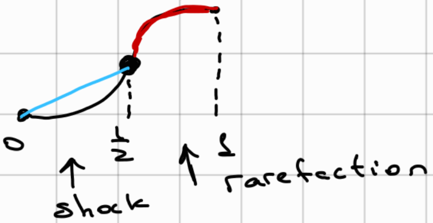
not genuinely nonlinear

not linearly degenerate

How does the solution look like?

The naive idea is that the solution consists of 2 parts (say, we are trying to solve the Riemann problem 

):



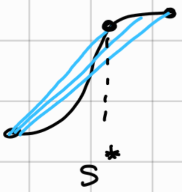
but this is a strange solution

?!

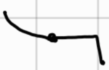
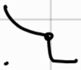
slope  = shock speed

tangents  = speed of S for a rarefaction wave

To avoid such situation, we see that



only $S \geq s^*$ are valid for left state of a shock. Here u^* is the abscissa of the tangent line from $u=0$ to graph of f

Then we get too many solutions:  .... 

On the other hand, may be not all the shocks satisfy additional entropy condition?

Let's consider the vanishing viscosity criterion

$$s_t + (f(s))_x = \varepsilon s_{xx}$$

We seek for solutions of the form

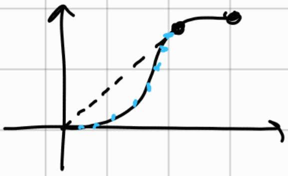
$$s = s\left(\frac{x-vt}{\varepsilon}\right) = s(\xi)$$

$$\Rightarrow \quad -v s' + (f(s))' = s''$$

Integrate from $\xi = -\infty$ till ξ :

$$-vs + f(s) = s'$$

$$\begin{cases} s' = f(s) - vs \\ s(-\infty) = 0 \\ s(+\infty) = s_R \end{cases}$$

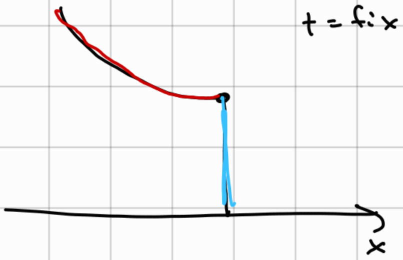


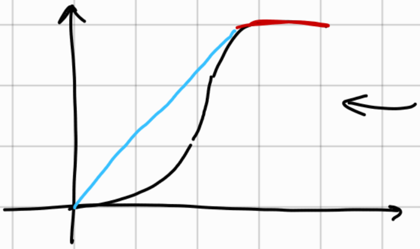
Necessary condition : $f(s) - vs < 0$

$f(s) < vs$.

This is valid only for points $s \le s^*$.

Thus, the only admissible shock wave is

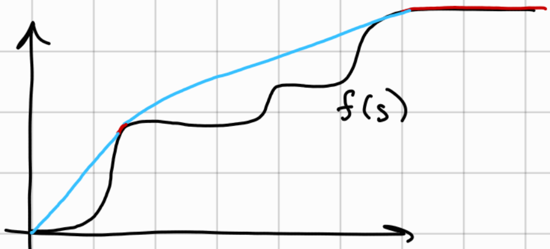




← convex hull of graph f

In general there is the following algorithm of constructing a solution to a Riemann problem

$$s(x,0) = \begin{cases} s_L , & x<0 \\ s_R , & x>0 \end{cases} \qquad s_L > s_R$$

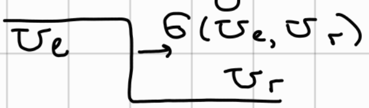


Take convex hull of f(s): say $\tilde{f}(s)$:

- if $\tilde{f}(s) = f(s)$, then this s is moving with $f'(s)$ (as a part of rarefaction wave)
- if $\tilde{f} > f$, then this corresponds to a shock wave

(*) $U_t + F(U)_x = 0$, $U \in \mathbb{R}^m$

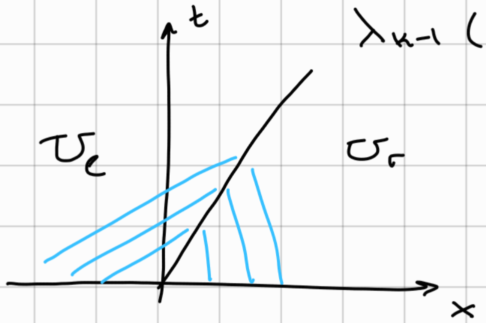
Entropy criteria for weak solutions for systems of conservation laws.



$$U(x,t) = \begin{cases} U_l, & x < \sigma t \\ U_r, & x > \sigma t \end{cases}$$

① **Lax**: $\exists k = 1 .. m$:

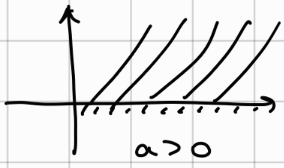
$$\lambda_k(U_r) < \sigma(U_l, U_r) < \lambda_k(U_l)$$
$$\lambda_{k-1}(U_l) < \sigma(U_l, U_r) < \lambda_{k+1}(U_r)$$



There is only one index k such that the shock speed σ is intermediate to the characteristic speed λ_k on both sides of the shock.

There exists an "empirical" explanation that the total amount of characteristics that "come" to shock should be (n+1).

] $u_t + au_x = 0$ in $x>0$, $t>0$:



we need to define $u(0,t)$

we do not need to define $u(0,t)$

] $U_t + AU_x = 0$ linear system

Let $P^{-1}AP = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$

Then for $V = P^{-1}U$ the system $v^i_t + \lambda_i v^i_x = 0$ decouples into m equations.

If k: $\lambda_i < 0$, $i \le k$

$\lambda_i > 0$, $i = k+1, \dots$

Thus we need $(n-k)$ conditions on the components of v (and u) on the bound. $x=0$

] More generally, if we have a boundary that moves with speed s ($s = 0$ was for quarter plane problem) and if

$$\lambda_1 < \dots < \lambda_k < s < \lambda_{k+1} < \dots < \lambda_m$$

we must give $(n-k)$ conditions on u in order to be able to specify the solution in the region $x - st > 0$, $t > 0$.

Now extend this to a discontinuity of a hyperbolic system: let $\lambda_1(u) < \lambda_2(u) < \dots < \lambda_m(u)$ be eigenvalues of $DF(u)$. Suppose,

$$\lambda_k(U_r) < s < \lambda_{k+1}(U_r)$$

on the right bound. of disc

Then we should specify $(n-k)$ conditions

Similarly, if we assume

$$\lambda_j(U_e) < s < \lambda_{j+1}(U_e)$$

then we must specify j conditions on the left boundary of disc.

We have $(n-1)$ jump conditions from RH:

$(n-k) + j = n-1$ or $j = k-1$. That are exactly the Lax conditions above.

② Vanishing viscosity (limit of traveling waves)

$$u^\varepsilon(x,t) = v\left(\frac{x - \sigma t}{\varepsilon}\right), \quad v : \mathbb{R} \to \mathbb{R}^m$$

Then v must solve ODE:

$$\begin{cases} \ddot{v} = -\sigma \dot{v} + \dot{(F(v))} \\ v(-\infty) = U_\ell, \quad v(+\infty) = U_r, \quad \lim\limits_{\xi \to \pm\infty} \dot{v} = 0 \end{cases}$$

Integrating we get:

$$\dot{v} = F(v) - F(U_l) - \sigma(v - U_l)$$

Now the sysmem is m-dimensional and in general more difficult

Thm (existence of traveling waves for genuinely nonlinear systems)

Assume (λ_k, r_k) is genuinely nonlinear for $k = 1 \dots m$. Let U_r be sufficiently close to U_l. Then there exists a travelling wave solution connecting U_l and U_r iff $U_r \in S_k^-(U_l)$ for some k

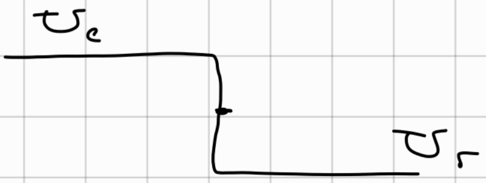
(without proof)

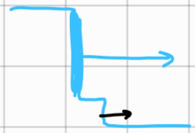
③ Liu criterion (internal stability of a shock)

Let $U_r \in S_k(U_l)$ for some $k = 1 \dots m$

and $\sigma(z, U_l) > \sigma(U_r, U_l) > \sigma(U_r, z)$

for each z lying on the curve $S_k(U_l)$ between U_r and U_l.





④ Entropy / Entropy-flux pair

Def: we say two smooth functions $\Phi, \Psi : \mathbb{R}^m \to \mathbb{R}$ comprise an entropy / entropy-flux pair for the conservation law $U_t + F(U)_x = 0$ provided: (a) Φ is convex

(b) $D\Phi(z) \cdot DF(z) = D\Psi(z)$, $z \in \mathbb{R}^m$

Rmk: if solution of $U_t + F(U)_x = 0$ is smooth, then $$D\Phi \cdot U_t + \underbrace{D\Phi \cdot DF}_{D\Psi} \cdot U_x = 0$$

$$\Rightarrow \quad \Phi(U)_t + \Psi(U)_x = 0$$ – this is just an additional conservation law!

For shocks we do not have this equiality, but instead we could replace $\Phi_t + \Psi_x = 0$ with inequality: $\Phi(U)_t + \Psi(U)_x \le 0$, $\begin{matrix} t>0 \\ x\in\mathbb{R} \end{matrix}$

In applications, Φ sometimes is negative of physical entropy and Ψ is entropy flux (this explains the terminology)

The rigorous understanding of the inequality in weak sense: $\forall\ \varphi \in C_c^\infty(\mathbb{R}\times(0,+\infty))$, $\varphi \geq 0$:

$$\text{(EEF)} \quad \int_0^\infty \int_{\mathbb{R}} \left(\Phi(U)\varphi_t + \Psi(U)\varphi_x \right) dx\,dt \geq 0$$

Def: we call $U \in \mathbb{R}^m$ an entropy solution of $(*)$ provided U is a weak solution of $(*)$ and satisfies inequalities (EEF) for every entropy / entropy-flux pair (Φ, Ψ)

Why such inequality? This can be easily seen if we think of u as a limit of U^ε – solution of vanishing viscosity method:

$$U^\varepsilon_t + F(U^\varepsilon)_x = \varepsilon U^\varepsilon_{xx} \quad \Big| \cdot D\Phi(U^\varepsilon)$$

$$\Phi(U^\varepsilon)_t + \Psi(U^\varepsilon)_x = \varepsilon D\Phi(U^\varepsilon) U^\varepsilon_{xx}$$

$$D\Phi(U^\varepsilon)\,U^\varepsilon_{xx} = \left(\Phi(U^\varepsilon)\right)_{xx} - \left(D^2\Phi(U^\varepsilon)U^\varepsilon_x\right)U^\varepsilon_x$$

Φ - convex $\Rightarrow$ $\left(D^2\Phi(U^\varepsilon)\cdot U^\varepsilon_x\right)U^\varepsilon_x \geq 0$

$$\Phi(U^\varepsilon)_t + \Psi(U^\varepsilon)_x \leq \varepsilon\left(\Phi(U^\varepsilon)\right)_{xx} \qquad \forall\varepsilon$$

$$\underline{\leq 0}$$

When we can find an entropy ?

Example: $\boxed{m=1}$ scalar conservation law for any convex Φ we can find a flux function:

$$\Psi(z) = \int_{z_0}^{z} \Phi'(w)\,F'(w)\,dw \quad , z\in\mathbb{R}$$

With this notion of entropy solution one can prove that there exists at most 1 weak solution of a scalar conservation law

$\boxed{m=2}$ p-system: $(\Phi_{z_1}, \Phi_{z_2})\begin{pmatrix} 0 & -1 \\ p'(z_1) & 0 \end{pmatrix} = \begin{pmatrix} \Psi_{z_1} \\ \Psi_{z_2} \end{pmatrix}$

$$\begin{cases} p'(z_1)\,\Phi_{z_2} = \Psi_{z_1} \\ -\Phi_{z_1} = \Psi_{z_2} \end{cases}$$

$$\Rightarrow \begin{cases} \Phi = \frac{z_2^2}{2} - \int_0^{z_1} p(w)\,dw \\ \Psi = p(z_1)\,z_2 \end{cases}$$

— convex as $p'<0$

check!!!

Lecture 15 : Reaction-diffusion equations

$u = u(t,x)$, $x \in \mathbb{R}^N$, $t > 0$, $u \in \mathbb{R}^m$

$$(*) \quad \partial_t u \; \underbrace{- \Delta u}_{\text{(local) diffusion}} = \underbrace{f(u)}_{\text{reaction term}}$$

- excitible medium : more generally $f = f(t,x,u)$
- Δu — comes from particles moving according to Brownian motion (in a rough way, the population tends spread out uniformly, to move towards areas where there are fewer individuals)

"Intuitive" probabilistic justification :

Let the population consist of finite number n of individuals. Consider a discrete space:

$$\{ \lambda k : k \in \mathbb{Z}^N \} \subset \mathbb{R}^N , \qquad \lambda > 0$$

For a given individual we denote :

$p(t,x)$ – probability that the individual is at point x at time t.

$$X_k(t,x) = \begin{cases} 1 , & \text{if } k\text{-th individual is at point } x \text{ at time } t \\ 0 , & \text{otherwise} \end{cases}$$

Then $\quad U(t,x) = \frac{1}{n} \sum_{k=1}^{n} X_k(t,x)$ - normalized distrib. of the population

Assuming the movements of individuals are independent of each other, $U(t,x) \to p(t,x)$.

At each instant an individual can:

— move to a neighbouring point with prob. $q < \frac{1}{2n}$

— do not move with probability $1 - q \cdot 2n$

Note that the probability q does not depend on the position in time and space, nor on the previous position $\Rightarrow$ random walk $\Rightarrow$

$$p(t+\tau, \lambda k) = (1 - 2nq)\, p(t, \lambda k) + q \sum_{j=1}^{n} \left[p(t, \lambda(k+e_j)) + p(t, \lambda(k-e_j)) \right]$$

Assume that there exists a regular $p(t,x)$ for which the same relation is true for all x, t. So

$$\partial_t p + O(\tau) = \frac{q\lambda^2}{\tau} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial x_j^2} + O\left(\frac{\lambda^3}{\tau}\right)$$

Now let $\lambda, \tau \to 0$ such that $\frac{q\lambda^2}{\tau} \to D \in (0, +\infty)$

Thus, we get $\partial_t p = D \cdot \Delta p$.

Examples : ① population dynamics : u - concentration density
(ecology)

$$u_t - u_{xx} = f(u)$$

For a moment forget about diffusion and consider an ODE : $u_t = f(u)$, $u(0) = u_0$

Cases : ⓐ $f(u) = ru$ (Malthus equation '1798)

Solution: $u(t) = u_0 e^{rt}$, $r \in \mathbb{R}$

r - growth rate, the population grows infinitely (which is not natural)

ⓑ $f(u) = ru\left(1 - \frac{u}{k}\right)$ (logistic equation, ~ 1838)

$r \in \mathbb{R}$, $k \in \mathbb{R}$

Explicit solution: $u(t) = \dfrac{k}{1 + \left(\frac{k}{u_0} - 1\right) e^{-rt}}$

We observe, that :

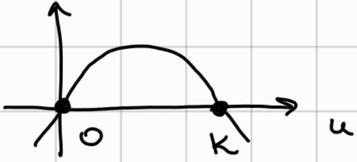
(i) whenever $u_0 > 0$, the solution is well-defined for $\forall t > 0$, $u(t) > 0$ and $u(t) \xrightarrow[t \to +\infty]{} k$

(ii) $u_0 = 0 \Rightarrow u(t) \equiv 0$

This corresponds to a more general fact that we will see later !

→ When u increases, there is a competition for resourses. Here k is called the capacity of environment

More general : monostable equations : $\dot{u} = f(t,u)$



assumptions: $f(0) = f(K) = 0$, f - Lipch in u

$f > 0$ for $u \in (0,K)$

$f < 0$ for $u \notin [0,K]$

Sometimes, there is an extra assumption : $\frac{f(u)}{u} \downarrow$

Lemma : $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ - continuous, loc. Lipschitz in u

(i) If $f(t,0) = 0 \ \forall t$, then if $u(0) > 0 \Rightarrow u(t) > 0 \ \forall t$

(ii) If u, v - two solutions and $u(0) > v(0)$, then $u(t) > v(t)$ (in the domain where both sol. exist)

(iii) If $u' \leq f(t, u(t))$ and $v' > f(t, v(t))$ and $u(0) \leq v(0)$, then $u(t) < v(t) \ \forall t$.

Rmk 1: when u satisfies the differential inequality $u' \leq f(t, u(t))$ we say that u is a sub-solution ; otherwise super-solution

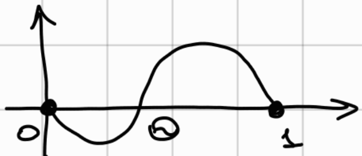
Rmk 2 : these statements are true for a single equation, but in general are not true for systems of eqs.

Rmk 3 : items (ii) and (iii) are the so-called "comparison theorems" in this very simple setting. We will see more of them for reaction-diffusion eqs.

Here $u = 0$ is unstable equilibrium point (asymp)
$u = K$ is stable equilibrium point (asymp)

Thus, the name "monostable" (1 stable point)

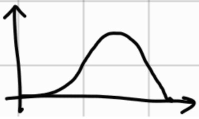
ⓒ $f(u) = u(1-u)(u-\theta)$, Bistable equations



strong Allee effect

or more general assumptions:
- $f(0) = f(\theta) = f(1) = 0$
- $f > 0$ for $u \in (\theta, 1)$
- $f < 0$ for $u \in (0, \theta)$

Weak Allee effect :



monostable equation without condition $\frac{f(u)}{u}$ is decreasing

Theorem: for $u(0) \in [0,1]$ the equation admits global-in-time solution $u(t) \in [0,1]$ $\forall t \in \mathbb{R}$

Moreover, if $u(0) < \Theta \Rightarrow u(t) \xrightarrow[t\to+\infty]{} 0$

$u(0) > \Theta \Rightarrow u(t) \to 1$

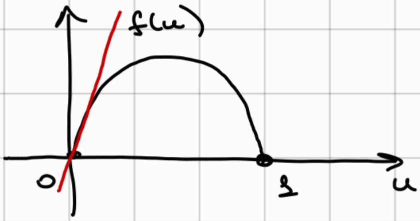
(the small population will turn off – may be not enough sexual partners or can not form big enough groups for fighting against predators)

This theorem explains the term "bistable":

$u=0$ and $u=1$ are stable equilibrium state

$u=\Theta$ - unstable equilibrium state

Concluding: we will consider 3 different $f(u)$:

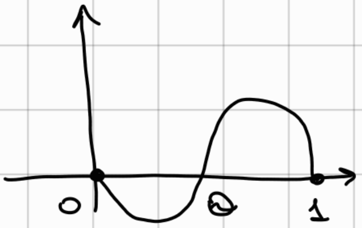


F-KPP

Fisher, Kolmogorov

Petrovskii, Piskunov (1937)

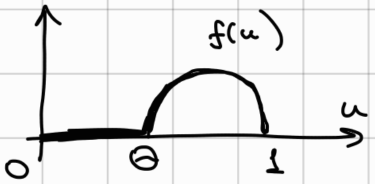
Monostable



Bistable

- monostable case with condition that $f(u)$ lies below the tangent line at $u=0$ (think of $f(u)=u(1-u)$)

There is also a case of ignition / combustion non-linearity: $f(u)=0$, $u \in [0,\theta]$



Rmk: there is one more notion of stability: linear stability

state α is called linearly stable if $f'(\alpha)<0$

state α —//— linearly unstable if $f'(\alpha)>0$

Thm: $f \in C^1$ in the vicinity of α ($f(\alpha)=0$)

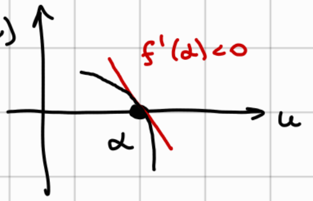
(i) If $f'(\alpha)<0$ and $u(0)$ is sufficiently close to α, then $u(t) \to \alpha$ as $t \to +\infty$

(ii) If $f'(\alpha)>0$, then no solution (except $u\equiv\alpha$) converges to α as $t\to+\infty$.

On the other hand, if $u(0)$ is close enough to α, then $u(t)\longrightarrow-\alpha$ as $t\to-\infty$.

Proof:

(i)



$f(u)>0$ for $u\in[\alpha-\varepsilon,\alpha)$

$f(u)<0$ for $u\in(\alpha,\alpha+\varepsilon]$

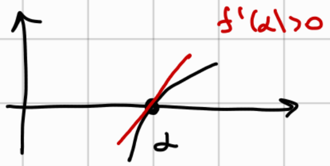
$\dot{u}=f(u)$

If $u(0)<\alpha \Rightarrow u(t)<\alpha$ and $\uparrow$

If $u(0)>\alpha \Rightarrow u(t)>\alpha$ and $\downarrow$

$u(t)$ can converge only to α.

(ii)



$f(u)<0$ for $u\in[\alpha-\varepsilon,\alpha)$

$f(u)>0$ for $u\in(\alpha,\alpha+\varepsilon]$

$\dot{u}=f(u)$

If $u(0)<\alpha \Rightarrow \dot{u}=f(u)<0 \Rightarrow u\downarrow$ and $u(t)<u(0)<\alpha$

If $u(0)>\alpha \Rightarrow \dot{u}=f(u)>0 \Rightarrow u\uparrow$ and $u(t)>u(0)>\alpha$

There are many-many ways to generalize these equations:

$\boxed{\Delta u} \rightsquigarrow \int_\Omega K(x-y)\,u(y)\,dy$ – non-local diffusion

$\rightsquigarrow$ general (uniformly elliptic) term $\sum_{i,j=1}^{N} a_{ij}(t,x)\,\partial_i\partial_j u$

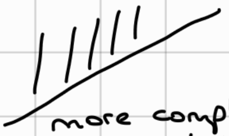
with condition $\exists\ 0<\alpha<\beta<+\infty$:

$\forall \xi\in\mathbb{R}^N, \forall t>0, \forall x\in\Omega \quad \alpha\|\xi\|^2 \le \sum a_{ij}(t,x)\,\xi_i\xi_j \le \beta\|\xi\|^2$

$\boxed{f(u)} \rightsquigarrow f(t,x,u)$ – depend on space x and time t

$\boxed{u\in\mathbb{R}} \rightsquigarrow \vec{u}\in\mathbb{R}^n$ – many species (Lotka-Volterra, predator-prey system, competitive media)

$\boxed{\Omega\subset\mathbb{R}^N} \rightsquigarrow$



more complex geometries

line of "fast" diffusion ("roads" in forests)

etc....

Other contexts: → combustion theory (propagation of flame, thermo-diffusive model)

→ probability (BBM - Branching Brownian Motion Mckean representation)

→ statistical physics etc....

Reaction-diffusion eqs: problem statement

(*) $$\partial_t u = D \Delta u + f(t,x,u)$$

- $t \in (0,+\infty)$
- $D > 0$
- $u \in \mathbb{R}$ - scalar
- $f(u)$ is of one of the types above

- $x \in \Omega$ → $\Omega = \mathbb{R}^N$
  → $\Omega \subset \mathbb{R}^N$ - bounded, connected with reg. boundary

\+ Initial condition: $u|_{t=0} = u_0(x) \in C(\Omega) \cap L^\infty(\Omega)$

\+ Boundary conditions:

(Neumann) $\partial_n u = 0$ for $(t,x) \in (0,+\infty) \times \partial\Omega$

(Dirichlet) $u = 0$ for —//—

(Robin) $\partial_n u + qu = 0$ for —//—

Interpretations:

Neumann: no individuals cross the boundary (in any direction) ✓

Dirichlet: exterior of Ω is extremely unfavorable so population density is zero at boundary

Robin: there is a flow of individuals entering $(q>0)$ or leaving the domain $(q<0)$

We consider classical solution u which satisfies

$$(**) \begin{cases} u \in C^0([0,+\infty) \times \bar{\Omega}) \\ \partial_t u \in C^0((0,+\infty) \times \Omega) \\ \forall i \quad \partial_{x_i} u \in C^0((0,+\infty) \times \bar{\Omega}) \\ \forall i,j \quad \partial_{x_i x_j} u \in C^0((0,+\infty) \times \Omega) \end{cases}$$

equation (*), initial and one of the boundary cond

If $\Omega = \mathbb{R}^N$ we also assume some growth cond. at infinity: $\forall T > 0 \quad \exists A, B > 0$:

$$|u(t,x)| \le A e^{B|x|}, \quad x \in \mathbb{R}^N, \quad t > 0$$

What are the important topics?

① Comparison theorems: roughly speaking if $u(0,x) \le v(0,x)$ are both solutions of (*) then $u(t,x) \le v(t,x)$ $\forall t>0$

Closely connected to maximum principle for parabolic PDEs.

This can be very helpful:

example 1: $u_t = \Delta u + u(1-u)$

$u(0,x) \in [0,1]$ $\forall x \in \mathbb{R}^N$

- $u \equiv 0$ is solution and $u(0,x) \ge 0$
  $\Rightarrow$ $u(t,x) \ge 0$
- $u \equiv 1$ is solution and $u(0,x) \le 1$
  $\Rightarrow$ $u(t,x) \le 1$

Thus, $u(0,x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$

example 2: $u_t = \Delta u - u^3$

$u|_{t=0} = u_0 \in [m, M]$, $x \in \Omega \subset \mathbb{R}^N$

Consider $\begin{cases} \dot{v} = -v^3 \\ v(0) = m \end{cases}$ and $\begin{cases} \dot{w} = -w^3 \\ w(0) = M \end{cases}$

These are sub and supersolutions:

$$v(t) \le u(x,t) \le w(t)$$

$$-\frac{dv}{v^3} = dt \Rightarrow \frac{1}{2v^2} - \frac{1}{2m^2} = t \Rightarrow v = \left(\frac{1}{m^2} + 2t\right)^{-\frac{1}{2}}$$

$v(t) \to 0$ as $t \to \infty$

Analogously, $w(t) \to 0$ as $t \to \infty$

Thus, if u exists, then

$$\underset{\substack{\downarrow \\ 0}}{v(t)} \le u(x,t) \le \underset{\substack{\downarrow \\ 0}}{w(t)} \Rightarrow \underset{t\to+\infty}{u \to 0}$$

→ well-posedness of (*) : $\exists!$ cont. dependence

→ special solutions : traveling waves (planar)
take direction $e \in \mathbb{R}^N$ and consider a solution of the form:

$$u(t,x) = \tilde{u}(\overbrace{x\cdot e}^{\text{scalar product}} - vt)$$

$\tilde{u} : \mathbb{R} \to \mathbb{R}$

v - speed of propagation

We will see that for different nonlinearities there exist travelling waves (TW)

$x \in \mathbb{R}^1$: FKPP : $\exists\, c^*$: $\forall c \geq c^*$ $\exists$ TW

Bistable: $\exists!\, c$: $\exists$ TW

→ $x \in \mathbb{R}^1$: long-time behaviour as $t \to +\infty$ for some initial data (like  Heaviside) the solution u of (*) "converges" to a $\boxed{\text{TW}}$

§ Maximum principle for parabolic equations

This is an extension of the results that we have seen for ODEs. First, some definitions:

Def 1: $u(t,x)$ is called sub-solution of (*) if it satisfies (**) and inequalities:

$$\partial_t u \leq \Delta u + f(t,x,u)$$

and on the boundary (if applicable): on $\partial\Omega$

(Neumann) $\partial_n u \leq 0$; (Dirichlet) $u \leq 0$; (Robin) $\partial_n u + gu \leq 0$

If $\Omega = \mathbb{R}^N$, then $|u| \leq Ae^{B|x|}$, $A, B > 0$

Analogously, $v(t,x)$ is called a super solution if all inequalities are reversed (except $|v| \le A e^{B|x|}$)

We want to prove the following theorem:

Theorem (comparison principle)

Let u and v be sub- and super-solutions of the reaction-diffusion eq $(*)$.

(i) If $u(0,x) \le v(0,x)$ for $x \in \bar{\Omega}$, then $u(t,x) \le v(t,x)$ for $t>0$, $x \in \bar{\Omega}$

(ii) If moreover, $u(t_0,x_0) = v(t_0,x_0)$ for some $t_0 > 0$, $x_0 \in \Omega$, then $u \equiv v$.

(iii) If Ω is bounded and the boundary condition is of Neumann or Robin type, then (ii) is true even for $x_0 \in \partial\Omega$

Note that the difference $(u-v)$ satisfies

$$\partial_t (u-v) \le \Delta(u-v) + f(t,x,u) - f(t,x,v)$$

Thanks to regularity of u, v, f we can rewrite this equation as follows: $w = u - v$

$$(1) \quad \partial_t w \le \Delta w + g(t,x) w$$

where

$$g(t,x) = \begin{cases} \dfrac{f(t,x,u) - f(t,x,v)}{u-v} & \text{if } u \ne v \\ \partial_u f(t,x,u) & \text{if } u = v. \end{cases}$$

is continuous and uniformly bdd function

So we reduced a problem to studying the linear eq (1) and showing $w \le 0$ $\forall\, t>0$, $x \in \bar{\Omega}$.

Linear problem and maximum principle

Let us consider a more general case:

(2) $\partial_t u = \Delta u + \sum b_i(t,x)\, \partial_i u + c(t,x)\, u$

Let b_i, c be uniformly bdd.

Thm 1 (weak maximum principle)

(i) Let be a sub-solution of linear eq (2).
If $u(0,x) \leq 0$, then $u(t,x) \leq 0$ $\forall t>0$

(ii) Let v be super-solution of linear eq (2).
If $v(0,x) \geq 0$, then $v(t,x) \geq 0$ $\forall t>0$.

because $u(x_0,t_0)=0 \Rightarrow u\equiv 0$

Thm 2 (strong maximum principle)

(i) Let u be a subsolution of (2) and $u(0,x)\leq 0$.
If $\exists\, t_0>0, x_0\in\Omega : u(t_0,x_0)=0 \Rightarrow u\equiv 0$ on $[0,t_0]\times\Omega$

(ii) Let v be a supersolution of (2) and $v(0,x)\geq 0$.
If $\exists\, t_0>0, x_0\in\Omega : v(t_0,x)=0 \Rightarrow v\equiv 0$ on $[0,t_0]\times\Omega$

(iii) If Ω is bdd, then for Neumann and Robin bc the same statement as in (i), (ii) are true if $x_0\in\partial\Omega$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v=-u$.

Proof of maximum principle:

▶ We will prove in 2 cases: ⓐ Ω - bdd, Dirichlet
ⓑ $\Omega=\mathbb{R}^N$

First, let's prove the simple case:

Lemma: let u be a subsolution with strict ineq:

$\partial_t u - \Delta u - \sum b_i(t,x)\partial_i u - c(t,x)\,u<0$, $u(0,\cdot)<0$, $\boxed{u|_{\partial\Omega}<0}$

$\Rightarrow u(t,x)<0$

Proof of lemma:

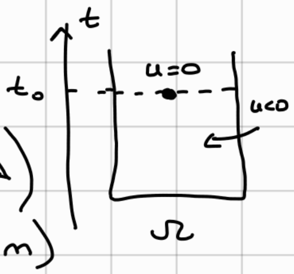
Indeed, take first time $t_0>0$ such that $u(x_0,t_0)=0$ for $x_0 \in \Omega$.

At this point: $\partial_t u \geq 0$

$\Delta u \leq 0$ (the local picture )

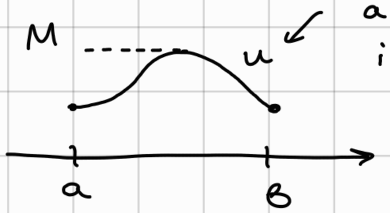
$\partial_i u = 0$ (as it is local maximum)

$u=0$



$\Rightarrow \partial_t u - \Delta u - \sum b_i \partial_i u - cu \geq 0$ (!?) ■

Lecture 16: Maximum principles for ODEs.



a non-constant function that achives its maximum over an interval $[a,b]$

Let $[a,b] \subset \mathbb{R}$

$u \in C^2((a,b)) \cap C^0([a,b])$

Consider a differential operator:

$$L = -\frac{d^2}{dx^2} + g\frac{d}{dx} + h$$

- g, h - bounded functions on (a,b)

Let

$$M = \max_{[a,b]} u$$

Question: how inequalities for Lu can lead to conclusions about M?

Lemma 1 (basic lemma for $h \equiv 0$):
Let $h \equiv 0$ and $Lu < 0$. Then u can equal to M only at the endpoints $x=a$ or $x=b$.

Proof:

▶ By contradiction: suppose $\exists x_0 \in (a,b)$: $u(x_0) = M$

Then $u'(x_0) = 0$

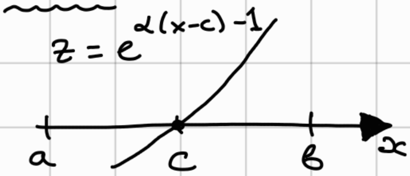
$u''(x_0) \leq 0$

$$L \Rightarrow Lu\Big|_{x_0} \geq 0 \quad (!?) \quad ■$$

Thm 1 (one-dimensional maximum principle for $h \equiv 0$)
Let $h \equiv 0$ and $Lu \leq 0$.
Then if $\exists c \in (a,b)$: $u(c) = M \Rightarrow u \equiv M$.

Proof:

▶



Suppose $u \not\equiv M \Rightarrow$
$\exists d \in (a,b)$ such that
$u(d) < M$ (w.l.o.g. $d > c$)

We would like to construct a "barrier" $z(x)$ such that for $w = u + \varepsilon z$:

$$Lw < 0 \quad \text{on } (a,b)$$

and we could apply lemma 1.

Take

$$z = e^{\alpha(x-c)} - 1.$$

$z(c) = 0$, $z > 0$ for $x \in (c, b)$

$$Lz = (-\alpha^2 + g\alpha)\, e^{\alpha(x-c)}$$

Since g is bounded we can choose $\alpha > 0$ large enough such that $Lz < 0$

Thus, $Lw = Lu + \varepsilon Lz < 0$.

Moreover, $w(a) = u(a) + \underset{\overset{\wedge}{0}}{\varepsilon z(a)} < u(a) \le M$

$$w(d) = \underset{\hat{M}}{u(d)} + \varepsilon z(d) < M$$

by taking very small ε we can guarantee that $w(d) < M$

Thus, we have a contradiction with Lemma 1. So, $u \equiv M$. ∎

Rmk: this idea of "adding a small barrier" is very useful and we will encounter this many times in future.
The choice of z is not unique!

Thm 2 (one-dimensional Hopf lemma for $h \equiv 0$)
Let $h \equiv 0$ and $Lu \le 0$.
If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$
Similarly, if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$

Rmk: the essense of the Hopf lemma is in strict inequality $u'(a) < 0$. Because the non-strict inequality is straight forward: if $u(a) = M \Rightarrow u'(a) \le 0$. So if the maximum is on the boundary, this point can not be a critical point (unless $u \equiv$ constant)

Proof:

Let $u(a) = M$ and by contradiction $\exists\, d \in (a,b) : u(d) < M$

We can use the same "barrier"

$$z = e^{\alpha(x-a)} - 1$$

and consider $w = u + \varepsilon z$.

First, $Lw < 0$ for sufficiently large α.

And $w(a) = M > w(d)$ for sufficiently small ε.

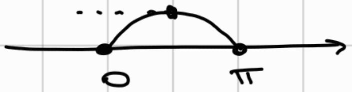
So w achieves its maximum at $x = a$.

$$w'(a) = u'(a) + \varepsilon\alpha \le 0$$

$$\Rightarrow u'(a) \le -\varepsilon\alpha < 0.$$ ■

Interestingly, if we relax condition $h \equiv 0$, the statements are no longer valid. Consider the following counter-example:

- $Lu = -u'' - u$
  Take $Lu = 0$



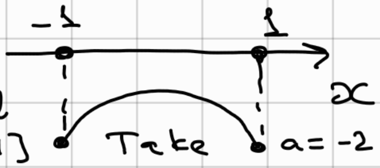
$\sin(x)$!

- $Lu = -u'' + u$, $x \in [-1,1]$

$u = -x^2 + a, a \in \mathbb{R}$

Look for the solution of the form

$Lu = 2 - x^2 + a \le 0$, $a \le x^2 - 2$ $\forall x \in [-1,1]$



Take $a = -2$

In these examples $h \cdot M \le 0$. If $h \cdot M \ge 0$, then everything ok!

Thm 3 (one-dimensional maximum principle for $h \not\equiv 0$)

exercise

Let $h \geq 0$ and $M \geq 0$.

If $Lu \leq 0$ on (a, b)

then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv M$.

Rmk: this theorem should also work for $h \leq 0, M \leq 0$

Thm 4 (one-dimensional Hopf lemma for $h \geq 0$)

exercise

Let $h \geq 0$.

Let $Lu \leq 0$ on (a, b) and $M \geq 0$.

If $u(a) = M$, then either $u'(a) < 0$ or $u \equiv M$.

Similarly, if $u(b) = M$, then either $u'(b) > 0$ or $u \equiv M$.

Thm 5 (comparison principle)

Let $h \equiv 0$, $f \in C^1$

$$Lu \leq f(x) \qquad x \in (a, b)$$

$$Lv \geq f(x), \qquad x \in (a, b)$$

Then if $\begin{cases} u(a) \leq v(a) \\ u(b) \leq v(b) \end{cases}$, then $u(x) \leq v(x)$ $\forall x\ (a, b)$

Moreover, if $\exists x_0 : u(x_0) = v(x_0) \Rightarrow u \equiv v$

Proof:

Γ▶ $w = u - v$; $\left.\begin{array}{l} Lw \leq 0 \\ w(a) \leq 0 \\ w(b) \leq 0 \end{array}\right\} \Rightarrow$ $w(x) \leq 0$ as maximum is obtained on the boundary $\begin{bmatrix} x = a \\ x = b \end{bmatrix}$

$\llcorner$ And if $w(x_0) = 0$ fo some $x_0 \in (a, b) \Rightarrow w \equiv 0$ ■

Rmk: if $f = f(x, u)$ the theorem does not easily work without any other assumptions

Rmk: The above strong max. principles say that subsolution u and supersolution v can NOT touch at a point: either $u \equiv v$ or $u < v$

This "untouchability" condition can be very helpful. Consider such an example.

Example: consider a boundary value problem:

$$(1) \begin{cases} -u'' = e^u, & x \in [0, L] \\ u(0) = u(L) = 0 \end{cases}$$

One can interpret the "u" as an equilibrium temperature: conditions $u(0) = u(L) = 0$ say that we have a "cold" boundary, while e^u is the "heating term". They compete with each other and non-negative solution corresponds to an equilibrium between these two effects. We would like to show that if the length of the interval L is suff. large, then no such equilibrium is possible. The physical reason is that the cold boundary is too far from the middle of the interval so that the heating term wins.

Task: show that for large enough $L > 0$ there is no non-negative solution of (1)

Step 1: consider $w = u + \varepsilon \Rightarrow$

$$(2) \begin{cases} -w'' = e^{-\varepsilon} e^{w} \\ w(0) = w(L) = \varepsilon \end{cases}$$

Step 2: consider family of functions:

$$V_\lambda(x) = \lambda \sin\left(\frac{\pi x}{L}\right)$$

They are solutions of the following problem:

$$(3)\begin{cases} -V_\lambda'' = \frac{\pi^2}{L^2} V_\lambda \\ V_\lambda(0) = V_\lambda(L) = 0 \end{cases}$$

Step 3: Notice that for L large enough

$$e^{-\varepsilon} e^{s} > \frac{\pi^2}{L^2} s \quad , \quad \forall s>0.$$

Thus, w as solution of (2) is a supersolution to (3) : $w(0)=w(L)=\varepsilon>0$

$$\begin{cases} -w'' \geq \frac{\pi^2}{L^2} w \\ w(0)=w(L) \geq 0 \end{cases}$$

We assume that $w \geq 0$.

Clearly, for small enough $\lambda>0$

$$V_\lambda(x) < w(x).$$

Step 4: (Sliding method) Now start increasing λ until some $\lambda_0>0$ s.t. the graphs of V_λ and w "touch" at some point:

$$\lambda_0 = \sup\{\lambda>0 : V_\lambda(x) \leq w(x) \,,\, 0 \leq x \leq L\}$$

Look at the difference: $p = V_\lambda - w$

- $-p'' \leq \frac{\pi^2}{L^2} p \qquad p(x) \leq 0$
- $p(0) = p(L) = -\varepsilon$

In addition, $\exists\, x_0 : p(x_0) = 0$. It can not be in (a,b) because of maximum principle and it can not be on the boundary (!?)

Exercise (for interest):

Show that $\exists L_1 > 0$ so that non-negative solution of (1) exists for all $0 < L < L_1$ and does not exist for all $L > L_1$.

Exercise (for now): consider

$$\begin{cases} -u'' - cu' = f(u), & x \in [-L, L] \\ u(-L) = 1, \ u(L) = 0 \end{cases}$$

Prove that if solution exists, then it is unique and decreasing ($u' < 0$)

Hint: use sliding method for 2 solutions u and v, e.g. consider

$$v_h(x) = v(x+h)$$

- Strong maximum principle for any h with assumption $M = 0$.

Thm 6 (one-dimensional maximum principle for $h \not\equiv 0$)

Let $M = 0$.

exercise

If $Lu \leq 0$ on (a, b), then u can attain maximum at some point $c \in (a, b)$ only if $u \equiv 0$.

Rmk: no assumptions on the sign of h!

Thm 7 (comparison principle):

$$f \in C^1$$

$$Lu \leq f(x, u) \qquad x \in (a, b)$$

$$Lv \geq f(x, u), \qquad x \in (a, b)$$

Then if $\begin{cases} u(x) \leq v(x) & \forall x \ (a, b) \\ \exists x_0 : & u(x_0) = v(x_0) \end{cases} \Rightarrow u \equiv v$

Lecture 17: Maximum principle for linear parabolic PDEs

Let us consider a linear parabolic PDE:

(1) $\partial_t u = \Delta u + \sum b_i(t,x)\,\partial_i u + c(t,x)\,u =: -Lu$

Here:
- $x \in \Omega$ (either bounded open connected set or $\mathbb{R}^N$, $N \geq 1$)
- $t \geq 0$
- $u : [0,+\infty) \times \Omega \to \mathbb{R}$ – scalar function
- coefficients b_i, c are continuous and uniformly bdd (=bounded)

Initial condition: $u(0,x) = u_0(x)$

Boundary conditions:
- Ω – bdd: (Dirichlet) $u|_{\partial\Omega} = 0$

  (Neumann) $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$

  (Robin) $\frac{\partial u}{\partial n} + q u\Big|_{\partial\Omega} = 0$
- $\Omega = \mathbb{R}^N$: $\exists\, A, B > 0$: $|u| \leq A e^{B|x|}$, $x \in \Omega$

Def: u - subsolution of (1) if $\partial_t u + Lu \leq 0$ and either $u|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} \leq 0$ or $\frac{\partial u}{\partial n} + qu\Big|_{\partial\Omega} \leq 0$

Analogously, v - supersolution if $\partial_t v + Lv \leq 0$ —//—

Thm 1 (weak maximum principle = weak MP)

(i) Let u be a subsolution of (1) s.t. $u(0,x) \leq 0$, Then $\forall t > 0$ $u(t,x) \leq 0$.

(ii) Let v be a supersolution of (1) s.t. $u(0,x) \geq 0$, Then $\forall t > 0$ $v(t,x) \geq 0$.

Rmk: it is clear that it is sufficient to consider subsolutions. For the supersolutions just consider $v = -u$.

Thm 2 (strong maximum principle = strong MP)

(i) Let u be a subsolution of (1) and $u(0,x) \leq 0$.
If $\exists\, t_0 > 0, x_0 \in \Omega : u(t_0, x_0) = 0 \Rightarrow u \equiv 0$ on $[0,t_0]\times\Omega$

(ii) Let v be a supersolution of (2) and $v(0,x) \geq 0$.
If $\exists\, t_0>0, x_0 \in \Omega : v(t_0, x) = 0 \Rightarrow v \equiv 0$ on $[0,t_0]\times\Omega$

(iii) If Ω is bdd, then for Neumann and Robin the same statement as in (i), (ii) are true if $x_0 \in \partial\Omega$.

Proof of maximum principle (weak and strong):

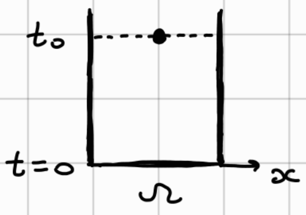
Case 1 : Dirichlet boundary conditions

Lemma 1 : Let $\partial_t u - Lu < 0$, $u(0,x) < 0$, $u|_{\partial\Omega} < 0$
Then $\forall\, t > 0$ $u(t,x) < 0$.

Proof

By contradiction. Let t_0 be the first time when $\exists\, x_0 \in \Omega : u(x_0,t_0) = 0$

At this point : $\partial_t u \geq 0$



$$-Lu \leq 0 \Leftarrow \begin{cases} \Delta u \leq 0 \\ \partial_{x_i} u = 0 \\ u = 0 \end{cases}$$

$$\Rightarrow \partial_t u + Lu \geq 0 \quad (!?)$$

Thus at $\forall\, x \in \Omega$, $t > 0$ $u(t,x) < 0$ ■

Observation : take $u = e^{kt} w$ for some $K \in \mathbb{R}$
$u < 0 \Leftrightarrow v < 0$ and $u \leq 0 \Leftrightarrow w \leq 0$
But now w satisfies :
$$\partial_t w - \Delta w - \Sigma\, b_i \partial_i w - (c-K) w \underset{(\leq)}{<} 0$$
Taking $K > \max |c|$ we can guarantee that $c - K < 0$, or taking $K < -\max|c|$ we have $c-k > 0$.

Let's take $K \geqslant \max |c| + 1$, and thus $c - k \leq -1$

In order not to change the notation we stay with letter "u" and consider $c \leq -1 < 0$ in (1).

Now we are ready to prove thm 1 (i).

By contradiction. Take the first moment $t_0 > 0$ s.t. $\exists\, x_0 \in \Omega$: $u(t_0, x_0) = \delta$ for some $\delta > 0$.

At this point (t_0, x_0) :

$$\left.\begin{matrix} \partial_t u \geq 0 \\ \Delta u \leq 0 \\ \partial_{x_i} u = 0 \end{matrix}\right\} \Rightarrow -Lu \leq c\delta \leq -\delta$$

$$\Rightarrow \quad \partial_t u + Lu \geq \delta > 0 \quad (!?)$$

Thus, for all $x \in \Omega$, $t > 0$ $\quad u(t,x) \leq 0$.

We have proven the weak MP for Dirichlet bc

Let's prove the strong maximum principle for Dirichlet bc.

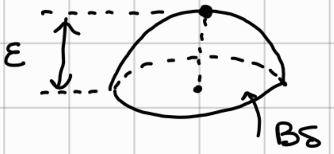
<u>Lemma 2</u>: Let u be subsolution of (1) with Dirichlet and $u(0,x) < 0 \quad \forall x \in \Omega \Rightarrow u(t,x) < 0 \quad \forall t > 0$

<u>Proof</u>:

▶ It is enough to consider $\Omega = B_\delta(0)$.

The idea is to construct a "barrier"

Let $w = u + \varepsilon(\delta^2 - |x|^2)^2 e^{-\alpha t}$



Take $\varepsilon > 0$ so small s.t. $w(0,x) < 0$. Moreover, $w\big|_{\partial B_\varepsilon} = u\big|_{\partial B_\varepsilon} \leq 0$

We can choose α such that w is a subsolution

Indeed, $\partial_i (\delta^2 - |x|^2)^2 = 2(\delta^2 - |x|^2) \cdot (-2x_i)$

$$\partial^2_{ii} (\delta^2 - |x|^2)^2 = -4(\delta^2 - |x|^2) + 8x_i^2$$

Then $(-L)(\delta^2 - |x|^2)^2 = (\Delta + \Sigma b_i \partial_i + c)(\delta^2 - |x|^2)^2 =$

$$= 8|x|^2 - 4N(\delta^2 - |x|^2) - 4 b \cdot x (\delta^2 - |x|^2) + c(\delta^2 - |x|^2)^2$$

By estimating $|B(t,x)| \le \|B\|_\infty$ and $|c(t,x)| \le \|c\|_\infty$ we obtain:

$$(\partial_t + L)\, z \le \varepsilon e^{-\alpha t} \Big[-\alpha \cdot (\delta^2 - |x|^2)^2 - 8|x|^2 + 4N(\delta^2 - x^2) +$$
$$+ 4|x| \cdot \|B\|_\infty (\delta^2 - |x|^2) + \|c\|_\infty (\delta^2 - |x|^2)^2 \Big]$$

We would like: $(\partial_t + L)\, z \le 0$.

<u>Naive idea</u>: just take $\alpha > 0$ very big and then the first term $-\alpha(\delta^2 - |x|^2)^2$ will be very negative and dominate all other (positive) terms.

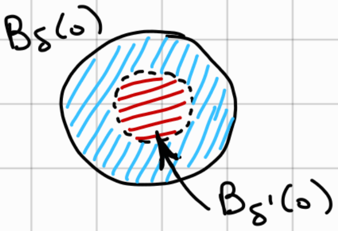
<u>Bad news</u>: the term $-\alpha^2(\delta^2 - |x|^2)^2$ is small close to the boundary of the $B_\delta(0)$. So the previous idea works only inside some smaller ball $B_{\delta'}(0) \subset B_\delta(0)$ $(0<\delta'<\delta)$

<u>What to do?</u> Divide the ball into 2 parts:

(1) $B_\delta(0) \setminus B_{\delta'}(0)$

(2) $B_{\delta'}(0)$

and estimate $(\partial_t + L)\, z$ in each part separately.



(1) If δ' is close to δ, then all terms that have $(\delta^2 - |x|^2)$ are small and the dominating term is $-8|x|^2$. Take δ' such that $\forall\ x \in B_\delta(0) \setminus B_{\delta'}(0)$ the following ineq is true

$$8|x|^2 > (\delta^2 - x^2) \cdot \big[4N + 4|x| \cdot \|B\|_\infty + \|c\|_\infty \cdot (\delta^2 - |x|^2) \big]$$

Or

$$8(\delta')^2 > (\delta^2 - (\delta')^2) \cdot \big[4N + 4\delta \|B\|_\infty + \delta^2 \cdot \|c\|_\infty \big]$$

Such δ' exists as $8(\delta')^2 \simeq 8\delta^2$ when $\delta' \simeq \delta$ and right hand side is almost 0.

Thus, for $x \in B_\delta \setminus B_{\delta'}$: $(\partial_t + L)\, z \le -\alpha \varepsilon e^{-\alpha t} (\delta^2 - |x|^2)^2 < 0$

(2) Now take α so big such that for all $x \in B_{\delta'}(0)$ we have:

$$\alpha\,(\delta^2-|x|^2)^2 > (\delta^2-|x|^2)\left[4N + 4\cdot|x|\cdot\|b\|_\infty + \|c\|_\infty(\delta^2-|x|^2)\right]$$

Divide by $\delta^2-|x|^2$ and it is enough to have

$$\alpha\cdot(\delta^2-(\delta')^2)^2 > \delta^2\left[4N + 4\delta'\cdot\|b\|_\infty + \|c\|_\infty\,\delta^2\right]$$

$$\alpha > \frac{\delta^2\left[4N+4\delta'\,\|b\|_\infty + \|c\|_\infty\,\delta^2\right]}{(\delta^2-(\delta')^2)^2}$$

(remember, here δ' is already some fixed value)

Thus, for $x \in B_{\delta'}(0)$: $(\partial_t + L)z < -8\,\varepsilon e^{-\alpha t}|x|^2 < 0$

$$\Rightarrow (\partial_t + L)\,w = \underset{\leq 0}{(\partial_t+L)\,u} + \underset{\leq 0}{(\partial_t+L)\left(\varepsilon\,(\delta^2-|x|^2)\,e^{-\alpha t}\right)} \leq 0$$

$$\Rightarrow \quad w \leq 0 \quad \Rightarrow \quad u < w \leq 0 \;;\; \text{q.e.d.}$$

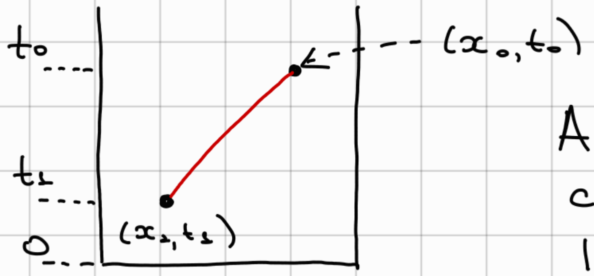
weak MP

Now let's finish proving the strong MP for (D).

Take (t_0, x_0) : $u(t_0,x_0) = 0$.

It is enough to prove that $u \equiv 0$ for $t\in[0,t_0]$, $x \in \Omega$

By contradiction, there exists a point (t_1, x_1), $t_1 < t_0$ such that $u(t_1,x_1) < 0$.



By continuity $u<0$ in $B_\delta(t_1,x_1)$ – ball in Ω

Assume that the segment connecting x_1 and x_0 in Ω lies in Ω (e.g. Ω-convex)

If necessary, take smaller δ s.t. $B_\delta(x) \subset \Omega$ for all x in this segment $[x_1, x_0]$ (this can be done by compactness of segment)

Now consider $w(t,x) = u\left(t,\ x + \frac{t-t_1}{t_0-t_1}\cdot(x_0-x_1)\right)$

$$\partial_t w = \partial_t u + \sum_{i=1}^{N} \partial_i u$$

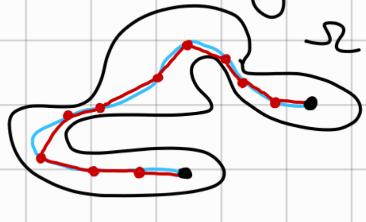
Clearly, w satisfies the equation of type (1)

By previous lemma : $w(t_1, x_1) = u(t_1, x_1)$

$w(t_0, x_1) = u(t_0, x_0)$

$w(t_1, x_1) < 0 \Rightarrow w(t_0, x_1) < 0 \Rightarrow u(t_0, x_0) < 0$ (!?)

It is easy to generalize this argument for arbitrary connected domains Ω, as there exists a path between x_1 and x_0 and this path can be approximated by segments. ■



Both weak and strong MP for Dirichlet bc are proven (case 1)

Case 2 : Neumann and Robin bc.

Lemma 3 (Hopf lemma)

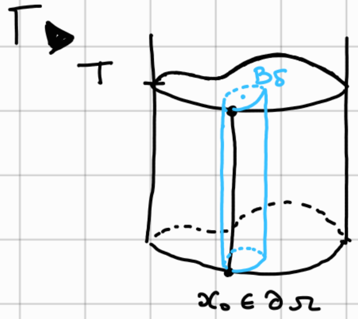
Let u be subsolution of (1) with NO boundary conditions. And let $u(t,x) < 0$ for all $t \in [0,T]$ and $x \in \Omega$.

If $u(T, x_0) = 0$ at $x_0 \in \partial\Omega$, then

$$\frac{\partial u}{\partial n}(T, x_0) > 0 .$$

Rmk : the sign $\frac{\partial u}{\partial n} \geq 0$ is clear, the important statement in lemma is STRICT inequal.

Proof :



By contradiction.

Let $\exists x_0 \in \partial\Omega$ s.t.

$$u(T, x_0) = \frac{\partial u}{\partial n}(T, x_0) = 0$$

Take a ball $B_\delta \subset \Omega$ s.t. $x_0 = \partial B_\delta \cap \partial\Omega$ (this is just some condition on regularity of $\partial\Omega$)

For simplicity we can always assume that the center of the ball B_δ is in the origin and the normal $n = (-1, 0, \dots 0)$

As $u<0$ in $\Omega \times [0,T]$, then $\forall\ 0<r<\delta$

$$\sup_{t\in[0,T]} \sup_{x\in B_r} u(t,x) < 0.$$

Consider

$$w = u + \varepsilon_1 (t-T) + \varepsilon_2 \left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2} \right]$$

$\alpha, \varepsilon_1, \varepsilon_2 > 0$ will be chosen soon.

We want to prove : for domain $A := B_\delta(0) \setminus \overline{B_r(0)}$

$0<r<\delta$

$$\begin{array}{cl} ⓐ & \left\{ \begin{array}{ll} \partial_t w + Lw \le 0, & x\in A,\ t\in[0,T] \\ w(0,x) < 0, & x\in A \\ w\big|_{\partial A}(t,x) \le 0 & \end{array} \right. \\ ⓑ & \\ ⓒ & \end{array}$$

Thus, by Dirichlet weak MP $\Rightarrow$ $w(T,x) \le 0$ for $x\in A$

This will be a contradiction with

$$w(T, -\delta, 0\dots 0) = u\Big|_{\substack{x=x_0\\ t=T}} = 0$$

$$\frac{\partial}{\partial n} w(T,\cdot) = -\partial_{x_1} w(T, -\delta, 0\dots 0) = -\partial_{x_1} u + \varepsilon_2 \alpha \cdot 2x_1 e^{-\alpha|x|^2}\Big|_{\dots} =$$

$$= 0 - \varepsilon_2 \cdot 2\alpha\delta \cdot e^{-\alpha\delta^2} < 0$$

Let's show ⓐ, ⓑ, ⓒ.

ⓐ $\partial_t w + Lw = \partial_t w - \Delta w - b\cdot\nabla w - cw \le$

$$\le \varepsilon_1 (1 + CT) - \varepsilon_2 e^{-\alpha|x|^2} \cdot \left[4\alpha^2 |x|^2 - 2N\alpha - 2C\alpha|x| - C \right]$$

where C is $\max(\|b_i\|_\infty, \|c\|_\infty)$.

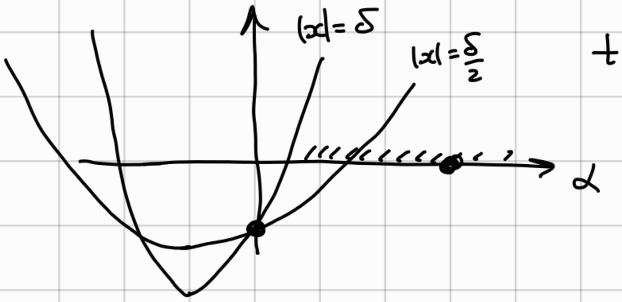
$$L\left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right] = \sum \frac{d}{dx_i}\left(-2\alpha x_i e^{-\alpha|x|^2}\right) + \sum b_i\left(-2\alpha x_i e^{\dots}\right)$$
$$+ c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right) = -2\alpha N e^{-\alpha|x|^2} + 4\alpha^2 \sum x_i^2 e^{-\alpha|x|^2}$$

$$+ \quad \Sigma' B_i \left(-2\alpha x_i e^{-\alpha|x|^2}\right) + c\left(e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right)$$

Fix $\alpha > 0$ s.t. $4\alpha^2|x|^2 - 2N\alpha - C\alpha|x| - C \geq \alpha$

for $x \in B_\delta \setminus B_{\delta/2}$: $d_1\alpha^2 + d_2\alpha + d_3 \geq 0$

This can be done if $|x|$ is not close to 0, e.g. $|x| > \delta/2$ (that's why we take the domain A to be a ring!)



Then w is a subsolution in A if

$$\text{(cond 1)} \quad \frac{\varepsilon_2}{\varepsilon_1} \geq \frac{(1+cT)e^{\alpha\delta^2}}{\alpha}$$

Ⓑ $$w(0,x) = u(0,x) \underbrace{- \varepsilon_1 T + \varepsilon_2 \left[e^{-\alpha|x|^2} - e^{-\alpha\delta^2}\right]_{<0}}_{\leq 0} \quad \text{for } x \in B_\delta \setminus \overline{B_r}$$

$$\text{(cond 2)} \quad \frac{\varepsilon_2}{\varepsilon_1} \leq \frac{T}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$$

If we choose r very close to δ, then

RHS of (cond 1) < RHS of (cond 2)

Ⓒ Boundary consists of 2 pieces: $\partial B_\delta, \partial B_r$

- Clear that $w(t,\partial B_\delta) = u(t,\partial B_\delta) + \varepsilon_1(t-T)$
  $\forall t \in [0,T]$ $\leq 0 + \varepsilon_1(t-T) \leq 0$

- $w(t,\partial B_r) = \underset{\overset{\wedge}{0}}{u(t,\partial B_r)} + \varepsilon_1(t-T) + \varepsilon_2\left(e^{-\alpha r^2} - e^{-\alpha\delta^2}\right)$

It is enough to take small $\varepsilon_2 > 0$, e.g.

$$\varepsilon_2 < \frac{-\sup_{t\in[0,\infty)} u(t,\partial B_r) \neq 0}{e^{-\alpha r^2} - e^{-\alpha\delta^2}}$$

L Then $w(t,\partial B_r) < 0$. ■

Next time we will finish the proof of the weak MP for Neumann / Robin bc.

Lecture 18: Today we will finish proving the maximum principles (weak and strong) for the Neumann, Robin b.c. and $\Omega = \mathbb{R}^N$ and briefly talk about the existence of the solutions to react.-diff. eqs.

Case 2 (Neumann, Robin b.c.)

W.l.o.g. $c < -1$.

- Want to prove: $u(0,x) \leq 0 \Rightarrow u(t,x) \leq 0$ $\forall t > 0$ $x \in \Omega$

By contradiction: $\exists \delta > 0$ and $\exists (t_0, x_0) : u(t_0, x_0) = \delta$ and t_0 is the first time when $u(t_0, x_0) = \delta$: for $0 \leq t < t_0$ $\forall x \in \overline{\Omega}$ $u(t,x) < \delta$.

(a) If $x_0 \in \Omega$, then the same argument as for Dirichlet case gives a contradiction:

$$(\partial_t + L) u \geq -cu \geq \delta > 0 \quad (!?)$$

(b) If $x_0 \in \partial\Omega$, we are in the context of the Hopf lemma for $w = u - \delta$.

Indeed, $w(t_0, x_0) = 0$ and $w(t,x) < 0$ if $\begin{cases} x \in \Omega \\ 0 \leq t \leq t_0 \end{cases}$ and w is a subsolution:

$$\partial_t w - \Delta w - \beta \cdot \nabla w - cw \leq -\delta c \leq 0$$

Thus, by Hopf lemma

$$\frac{\partial u}{\partial n}(t_0, x_0) = \frac{\partial w}{\partial n}(t_0, x_0) > 0$$

which contradicts the inequality $\frac{\partial u}{\partial n} \leq 0$ for the Neumann b.c.

This is a contradiction also for Robin b.c. as $\left(\frac{\partial u}{\partial n} + q u \right) \Big|_{(t_0, x_0)} > q u \Big|_{(t_0, x_0)} > q\delta > 0$

We assume $q > 0$ for the Robin b.c.

So we have proven the weak MP for (N), (R) ■

Let's prove the strong maximum principle for (N) and (R). As we already know $u(x,t) \leq 0$ $\forall$ $x \in \Omega$ and $x \in \partial\Omega$, we can apply the same argument as for the case of the Dirichlet b.c. In particular, if $u \not\equiv 0$, then $u<0$ $\forall t>0, x\in\Omega$ Apply the Hopf lemma again to see that $u < 0$ for $x \in \partial\Omega$, $t>0$. ■

<u>Case 3</u>: $\Omega = \mathbb{R}^N$. Take $w = u\varphi(x)$, where $\varphi \in C^\infty(\Omega)$ is strictly positive and

$$\frac{|\nabla\varphi|}{\varphi}, \quad \frac{|\Delta\varphi|}{\varphi} \in L^\infty(\mathbb{R}^N)$$

and

$$\varphi(x) = e^{-2B|x|} \qquad \text{for} \quad |x| >> 1.$$

$$w_t = u_t \varphi$$

$$\partial_i w = \partial_i u \cdot \varphi + \overset{w/\varphi}{\downarrow}{u} \cdot \partial_i \varphi$$

$$\partial^2_{ii} w = \partial_{ii} u \cdot \varphi + \partial_i u \cdot \partial_i \varphi + \overset{\frac{w}{\varphi}}{\downarrow}{u}\, \partial_{ii}\varphi$$

$$\Rightarrow \quad (\partial_t + L)u = (\partial_t + L)w - w \cdot \frac{\nabla\varphi \cdot B}{\varphi} - \nabla u \cdot \nabla\varphi - w \cdot \frac{\Delta\varphi}{\varphi}$$

Under the above conditions this operator is of the same type as for u, but for w we have:

$$|w| = |u\, e^{-2B|x|}| \leq A \cdot e^{-B|x|} \xrightarrow[\text{as } |x|\to\infty]{} 0$$

So the proof of the weak MP stays the same. The proof of the strong MP did not involve that Ω is bounded.

Well-posedness of reaction-diffusion eq.

$$(*) \quad \begin{cases} u_t = \Delta u + f(t,x,u) \\ u(0,x) = u_0(x) \\ + \text{ b.c.} \end{cases}, \quad \Omega \subseteq \mathbb{R}^N \begin{matrix} \nearrow \text{bdd, open} \\ \searrow \mathbb{R}^N \end{matrix}$$

Assume $f \in C^1(\mathbb{R})$, $u_0 \in C^0(\bar{\Omega})$.

① $\exists$ (existence)

② ! (uniqueness)

③ Continuos dependence on initial data

Thm (uniqueness of solution to react.-diff. eq.)
Let u, v be 2 solutions with the same initial conditions (D) or (N) or (R), then $u \equiv v$.

Proof: just by comparison theorem!

Thm (continuity of initial data)
Let u, v be 2 solutions with the same boundary conditions (D) or (N) or (R), but different initial data u_0, v_0. Then, $\forall t > 0$ $\exists K = \|\partial_u f\|_\infty$:

$$\|u(t,\cdot) - v(t,\cdot)\|_\infty \leq \|u_0 - v_0\|_\infty e^{Kt}$$

Proof

▶ $w = u - v$: $\quad \partial_t w - \Delta w = g(t,x) w$

g is uniformly bounded and $|g(t,x)| \leq K = \|\partial_u f\|_\infty$ and

$M e^{Kt}$ is a supersolution.

Taking $M = \|u - v\|_\infty$ we arrive at

$$u - v \leq \|u - v\|_\infty e^{Kt}$$

Analogously, $v - u \leq \|u - v\|_\infty e^{Kt}$ ■

Thm (continuous dependence on f)

Let $f_n \in C^1(\mathbb{R})$ and $f_n \to f$, $\partial_u f_n \to \partial_u f$ (uniformly)

Let u_n and u be solutions with reaction term f_n and f, respectively, and the same initial and boundary conditions.

Then $u_n \to u$ (locally uniformly in t)

Existence (only formulations and sketches of proof)

(1) linear case:

(1a) $\partial_t u = \Delta u + Ku$

Easy to pass to the heat equation by the change of variables: $u = e^{Kt} w$:

$$w_t = \Delta w$$

We know a lot about the heat eq.

(1b) non-homogeneous heat equation:

$$\begin{cases} \partial_t u = \Delta u + g(t,x) \\ u(0,x) = u_0(x) \\ \quad + \text{b.c. or boundness at } |x| \to +\infty \end{cases}$$

We can write an explicit formula for $\Omega = \mathbb{R}^N$

$$u(t,x) = \int_{\mathbb{R}^N} K(x-y,t)\, u_0(y)\, dy + \int_0^t \int_{\mathbb{R}^N} K(x-y, t-s)\, g(s,y)\, ds\, dy$$

where $K(x,t) = \dfrac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}$ - heat Kernel

② non-linear case: $\partial_t u = \Delta u + f(t,x,u)$

Let f, u_0 satisfy the following assumptions:

(U_0): $\exists M>0: \ |u_0| \leq M$

(F): $f \in C^1$, $f(t,x,0) \in L^\infty$, $\partial_u f \in L^\infty$

In particular,
- we can fix $K>1$, $\forall u \in \mathbb{R}$
  $|f(t,x,u)| \leq K(1+|u|)$.
- Moreover, $\forall u \geq -M$, $t \geq 0$, $x \in \Omega$
  $f(t,x,u) \leq K(1+u+M)$,
- and $\forall u \leq M$, $t \geq 0$, $x \in \Omega$,
  $f(t,x,u) \geq K(-1+u-M)$

Observation: we can always assume $\partial_u f > 0$ by using the following trick:

$$u(t,x) = e^{-Nt} \tilde{u}(t,x), \text{where } N = \sup|\partial_u f|$$

$$\partial_t u + Lu = f(t,x,u)$$

$$(\partial_t + L)\tilde{u} = \underbrace{N\tilde{u} + e^{Nt} f(t,x,e^{-Nt}\tilde{u})}_{\tilde{f}(t,x,\tilde{u})}$$

$$\partial_{\tilde{u}} \tilde{f} = N + \cancel{e^{Nt}} \partial_u f \cdot \cancel{e^{-Nt}} > 0.$$

So in this section (existence) we will assume $\boxed{\partial_u f > 0.}$

Thm (existence of solution to reaction-diff. eq)
Under the above conditions on Ω, u_0, f there exists a solution of $(*)$ for b.c. (D) or (N) or (R).

Idea: approximate the solution by a sequence of solutions $(u^k)_{k=1}^\infty$ of some linear probl. which solutions we already know.

(monotone iteration method)

Sketch of proof:

▶ First, consider $\underline{u}$ – the solution of the eq:

$$(\underline{U}) \quad \begin{cases} \partial_t \underline{u} - \Delta \underline{u} = K(-1 + \underline{u} - M) \\ \underline{u}(0,x) = u_0(x) \\ + \text{b.c.} \end{cases}$$

Solution exists (after change of variables we obtain just a heat equation)

Clearly, M is a supersolution of $(\overline{U}) \Rightarrow \underline{u} \leq M$

Thus, $K(-1 + \underline{u} - M) \leq f(t,x,\underline{u})$ by assumption (F).

Hence, $\underline{u}$ is a sub-solution of $(*)$

Analogously, consider $\overline{u}$ – the solution of

$$(\overline{U}) \quad \begin{cases} \partial_t \overline{u} - \Delta \overline{u} = K(1 + \overline{u} + M) \\ \overline{u}(0,x) = u_0(x) \\ + \text{b.c.} \end{cases}$$

Solution exists and $\overline{u}$ is a supersolution of $(*)$

Moreover, $\underline{u} < \overline{u}$ for $t > 0$ (by strong comp. thm)

Second, let's built an approximating seq.
Take $u^0 = \underline{u}$, and consider u^1 the solution of the following non-homogeneous heat eq:

$$\begin{cases} \partial_t u^1 - \Delta u^1 = f(t,x,u^0) \\ u^1(0,x) = u^0(x) \\ + \text{b.c.} \end{cases}$$

By comparison principle: $u^0 \leq u^1$.

Due to monotonicity of f:
$f(t,x,u^0) = f(t,x,\underline{u}) < f(t,x,\overline{u})$,
and comparison principle, we have

$$u^1 \leq \overline{u}$$

In total, we get: $\underline{u} = u^0 \leq u^1 \leq \overline{u}$.

Proceeding for $k = 1, 2, 3, \dots$ as follows:

$$\partial_t u^{k+1} - \Delta u^{k+1} = f(t,x,u^k)$$

we obtain
$$\boxed{\underline{u} = u^0 \leq u^k \leq u^{k+1} \leq \overline{u} \quad \forall k \in \mathbb{N}}$$

Third, at each point (t,x) the sequence converges $u^k(t,x) \to u(t,x)$.
We would like to pass to the limit in the equation and get:

$$\partial_t u - \Delta u = f(t,x,u).$$

Nevertheless, we know only that $u^k \to u$, but don't know the same result about the derivatives!

It would be enough to know that:

- $\|\partial_{x_i} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$
- $\|\partial_t u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$ (est.)
- $\|\partial_{x_i x_j} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq \tilde{C}$

for constant $\tilde{C}$ depending on τ, T, K

Here $C^{0,\alpha}([\tau,T]\times K)$ is a space of bounded and α-Hölder $(0<\alpha<1)$

τ compact set

continuos functions, that is $g \in C^{0,\alpha}$ means there exists a constant $C>0$:

$$|g(t_1,x_1) - g(t_2,x_2)| \le C\left(|t_1-t_2|^{\alpha} + |x_1-x_2|^{\alpha}\right)$$

supplied with the norm:

$$\|\cdot\|_{C^{0,\alpha}(\dots)} = \|\cdot\|_{L^\infty(\dots)} + C.$$

Why enough to know estimates (est.)?
Because of Arzela-Ascoli theorem:

> a set of functions f_n defined on a compact set, whose $C^{0,\alpha}$-norm is bounded, admits a subsequence which converges in C^0.

So by using (est.) and Arzela-Ascoli theorem, we can (several times) take a convergent subsequence, and pass to the limit in the equation. By uniqueness of the limit this is u that satisfies the reaction-diffusion eq.

Rmk: let us put under the carpet how to obtain estimates like (est.)
Sometimes they are called Schauder estimates and are based on fine properties of the heat kernel and the exact formula for solution: $u_t = \Delta u + g$

Just for the sake of completeness, let me give the formulation of Schauder-type estimates:

Thm (Schauder estimates):

means on every compact $[\tau,T]\times K \subset (0,\infty)\times\Omega$

Let $0<\alpha'<\alpha<1$, $g \in C^{0,\alpha}_{loc}((0,+\infty)\times\Omega)$.

Let u be a solution of

$$\begin{cases} u_t - \Delta u = g(x,t) \\ u(0,x) = u_0(x) \in C^0(\Omega) \end{cases}$$

Then:

- for all $0<\tau<T<+\infty$ and $\forall K \subset \overline{\Omega}$

$$\|u\|_{C^{0,\alpha}([\tau,T]\times K)} + \|\partial_{x_i} u\|_{C^{0,\alpha}([\tau,T]\times K)} \leq$$

$$\leq C \cdot \Big[\|g\|_{L^\infty([0,T+1]\times\overline{\Omega})} + \|u\|_{L^\infty([0,T+1]\times\overline{\Omega})}$$

- for all $0<\tau<T<+\infty$ and $\forall K' \subset K \subset \Omega$
  $K' \neq K$ - two compact sets:

$$\|\partial_t u\|_{C^{0,\alpha'}([\tau,T]\times K')} + \|\partial_{x_i x_j} u\|_{C^{0,\alpha'}([\tau,T]\times K')} \leq$$

$$\leq C \cdot \Big[\|g\|_{C^{0,\alpha}([\frac{\tau}{2},T+1]\times K)} + \|u\|_{L^\infty([0,T+1]\times K)}$$

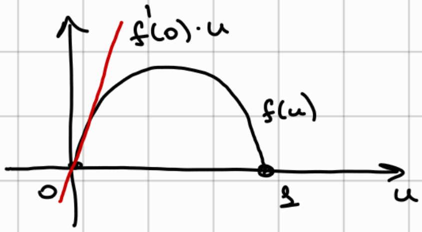
Here constant C may depend on τ, T, K, K', α.

Last comment on the proof of the thm $\exists$.
As $\underline{u}$ and $\overline{u}$ satisfy the initial cond. and $\underline{u} \leq u^k \leq \overline{u}$, then u^k will also satisfy the initial condition. $\blacksquare$

Lecture 19 : Existence of travelling wave (TW) solutions to reaction-diffusion eqs

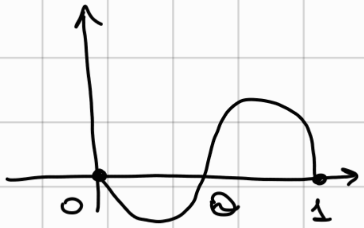
$$(*) \qquad u_t = \Delta u + f(u), \qquad u : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$$

Candidates for the reaction term f(u):



F-KPP
Fisher, Kolmogorov
Petrovskii, Piskunov (1937)

Monostable

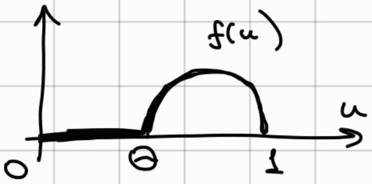


Bistable

- monostable case with condition that f(u) lies below the tangent line at u=0 (think of $f(u)=u(1-u)$)

$$f'(0) = \sup_{u \in (0,1]} \frac{f(u)}{u}$$

There is also a case of ignition / combustion non-linearity : $f(u) = 0,\ u \in [0,\theta]$



Consider $u(0,x) = u_0(x) \in [0,1] \Rightarrow u(t,x) \in [0,1]$ by comparison principle.

We are interested in traveling wave (TW) solutions (sometimes are also called traveling fronts = TF)

Fix direction $\vec{e} \in \mathbb{R}^N$ and consider the solution of the form: $\tilde{u} : \mathbb{R} \to [0,1]$ such that

$$(**) \qquad u(t,x) = \tilde{u}(x\cdot\vec{e} - ct), \quad c \in \mathbb{R} - \text{speed of TW (apriori unknown)}$$

Rmk 1: $\tilde{u}$ is constant on hyperplanes orthogonal to $\vec{e}$ and for this reason sometimes is called planar TW.

Rmk 2 : for simplicity of notation we will omit "~" and just write u istead of $\tilde{u}$

Putting form (**) into (*), we get an ODE:

$$(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1,\ u(+\infty) = 0\ ,\ u'(-\infty) = u'(+\infty) = 0 \end{cases}$$

Question : for which $c \in \mathbb{R}$ does there exist a solution of $(TW)_\infty$ problem ?

Thm (existence of TW solutions)

(i) In the bistable and combustion cases there exists a unique $c \in \mathbb{R}$ for which there exists a solution of $(TW)_\infty$.
Moreover, • u is unique and decreasing;
• sign of c coincides with the sign of $\int_0^1 f(u)\,du$.

(ii) In the monostable case $\exists\, c^* > 0$ such that there exists a solution (TW) iff $c \geq c^*$. When it exists the solution is unique and is decreasing.

(iii) In FKPP case $c^* = 2\sqrt{f'(0)}$.

Rmk : (i) in the bistable case if $c>0$, this means that the state one invades 0; if $c<0$ the state 0 invades 1; if $c=0$ there is a co-existence of two states.

(ii) The sign of speed c is easy to understand: multiply $(TW)_\infty$ by u' and $\int_{-\infty}^{+\infty} \cdot\cdot\, dz$:

$$-\int_{-\infty}^{+\infty} u''\cdot u' - c\int_{-\infty}^{+\infty} u'^2 = \int_{-\infty}^{+\infty} f(u)\underbrace{u'dz}_{du}\ \Big|\ \begin{matrix} -\infty \mapsto 1 \\ +\infty \mapsto 0 \end{matrix}$$

$$\cancelto{0}{-\frac{1}{2}(u')^2\Big|_{-\infty}^{+\infty}} \quad - c\int_{\mathbb{R}}(u')^2 = \int_1^0 f(u)\,du \Rightarrow \operatorname{sign}(c)=\operatorname{sign}\int_0^1 f$$

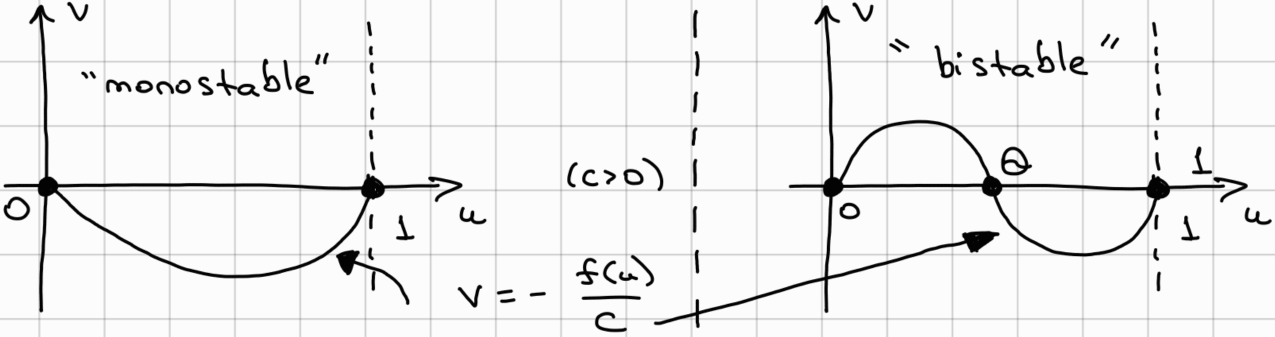
Proof:

▶ There exists 2 proofs: $\nearrow$ "dynamical" (phase plane method) $\searrow$ PDE

Sketch of "dynamical" proof

Write $(TW)_\infty$ as a system of two ODEs of first order: $u'=v$

$$\begin{cases} u'=v \\ v'=-cv-f(u) \end{cases} \qquad u\in[0,1]$$



Eq. $(TW)_\infty$ has a solution u $\Leftrightarrow$ $\exists$ heteroclinic orbit $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}$ such that $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}_{\xi\to-\infty} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$ and $\begin{pmatrix} u(\xi) \\ v(\xi)\end{pmatrix}_{\xi\to+\infty} = \begin{pmatrix} 0 \\ 0\end{pmatrix}$

Step 1: Zoom into vicinicity of fixed point:

$\begin{cases} u=1 \\ v=0 \end{cases}$ and $\begin{cases} u=0 \\ v=0 \end{cases}$ | $\begin{cases} u=0 \\ v=0 \end{cases}$ and $\begin{cases} u=\theta \\ v=0 \end{cases}$ and $\begin{cases} u=1 \\ v=0 \end{cases}$

Consider a linearized system at equilibrium point $(\alpha,0)$

$$\begin{cases} u'=v \\ v'=-cv-f'(\alpha)\cdot u \end{cases} \qquad \begin{pmatrix} u \\ v\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -f'(\alpha) & -c\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}$$

Eigenvalues are: $\begin{vmatrix} -\lambda & 1 \\ -f'(\alpha) & -c-\lambda \end{vmatrix} = \lambda^2+c\lambda+f'(\alpha)$

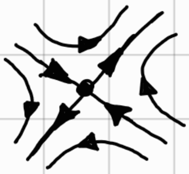
$$\lambda_\pm = \frac{-c\pm\sqrt{c^2-4f'(\alpha)}}{2} \quad ; \quad r_\pm = \begin{pmatrix} 1 \\ \lambda_\pm\end{pmatrix}$$

$\boxed{\begin{cases} u=1 \\ v=0 \end{cases}}$ $f'(1)>0 \Rightarrow \lambda_{\pm} \in \mathbb{R}$

In particular, $\lambda_+ > 0$ and $\lambda_- < 0$.

This is a saddle point.

Local behavior in the vicinity of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:



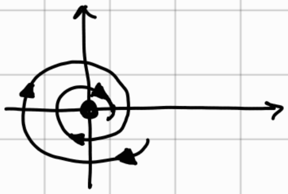
This picture is for the linearized system. By Grobman–Hartman theorem similar picture is true for the original nonlinear system.

Notice that there is exactly one orbit that leaves the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and our goal is to understand for which c it enters $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ without crossing $\{u=0\}$ (we want $u \in [0,1]$)

$\boxed{\begin{cases} u=0 \\ v=0 \end{cases}}$ Local behavior depends on the sign $(c^2 - 4f'(0))$, and is different for monostable and bistable cases

Case I : $\boxed{\text{monostable}}$

- If $0<c<2\sqrt{f'(0)}$, then $\lambda_{\pm} \in \mathbb{C} \setminus \mathbb{R}$ and this is a spiral point. This would immediately make $u<0$ at some point along the orbit. This is forbidden as $u \in [0,1]$.

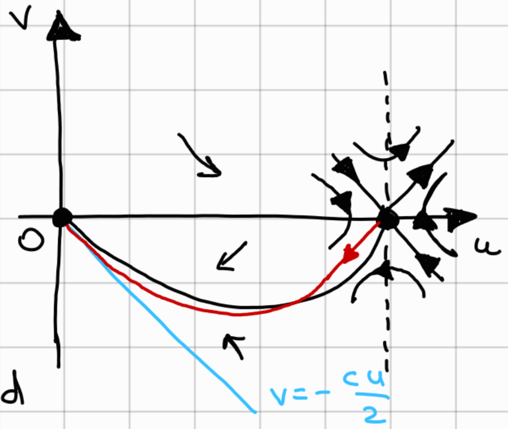


- So $c \geq 2\sqrt{f'(0)}$ (look at the statement for the FKPP case!)
- For $c > 2\sqrt{f'(0)}$ $\lambda_{\pm} < 0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a node
  For the FKPP case $f'(0) = \sup\limits_{u \in [0,1]} \frac{f(u)}{u}$.

Lemma: let $c > \sup\limits_{u\in(0,1]} \frac{f(u)}{u}$. Then the orbit $\binom{u}{v}$ s.t. $\binom{u}{v}\Big|_{\xi\to-\infty} = \binom{1}{0}$, does not intersect the line $v=-\frac{c}{2}u$ in the quarter plane $\{v<0\}\cap\{u>0\}$

Rmk: as a consequence we get that this orbit comes to point $\binom{0}{0}$ as $t\to\infty$ without crossing $u=0$.

Proof



▶ 1) At $t\to-\infty$ the "red" orbit is above $v=-\frac{cu}{2}$

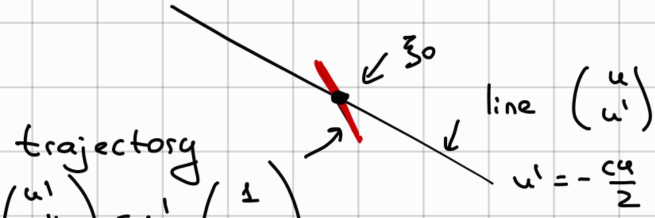
2) By contradiction: $\exists\, \xi_0\in\mathbb{R}$ – the first point of intersection of $v=-\frac{cu}{2}$ and the "red" orbit

At this point we have $\begin{cases} -u''(\xi_0) - cu'(\xi_0) = f(u(\xi)) \\ u'(\xi_0) = -\frac{c}{2}u(\xi_0)\end{cases}$

For $c > 2\sqrt{\sup \frac{f(u)}{u}}$, we obtain

$$u''(\xi_0) = -cu'(\xi_0) - f(u(\xi_0)) > -\frac{cu'(\xi_0)}{2}$$

This is a contradiction because this means that $\binom{u}{v}$ was already under the line $v=-\frac{cu}{2}$ for $\xi<\xi_0$.



line $\binom{u}{u'} = \binom{\xi}{-\frac{c\xi}{2}} \Rightarrow \binom{u'}{u''} = \binom{1}{-\frac{c}{2}}$

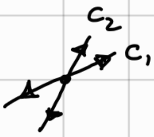
$$\frac{d}{d\xi}\binom{u}{u'} = \binom{u'}{u''} = u'\binom{1}{\underset{-\frac{c}{2}}{\underset{\vee}{a}}}$$

Thus the angle as $\xi<\xi_0$ is smaller as in the picture.

Thus, in a monostable case there exists at least 1 necessary orbit

In fact, we can say more. There is some monotonicity argument in how trajectories depend on c. Here are 2 observations:

Observation 1: locally in the vicinity of point $\binom{1}{0}$ the trajectory $\binom{u_1}{v_1}$ for c_1 is above the trajectory $\binom{u_2}{v_2}$ for c_2 if $c_1 > c_2$



Indeed, their tangent vector is $\begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$ and $\lambda_+ = \dfrac{-c + \sqrt{c^2 - 4 f'(1)}}{2}$ is a decreasing function of c.

Observation 2: in fact, these two trajectories for $c_1 > c_2$ do not intersect in the whole strip $\{u' < 0\} \cap \{ 0 < u < 1\}$

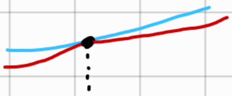
By contradiction, assume they intersect at some point (say, $\xi = 0$)

$$\begin{cases} u_1(0) = u_2(0) \\ u_1'(0) = u_2'(0) \end{cases}$$

This means:

$$u_1''(0) = -c_1 u_1'(0) - f(u_1(0)) > -c_2 u_2'(0) - f(u_2(0)) = u_2''(0)$$

$$\frac{d}{d\xi}\begin{pmatrix} u_1 \\ u_1' \end{pmatrix}\Big|_0 = \begin{pmatrix} u_1'(0) \\ u_1''(0) \end{pmatrix}$$



$$\frac{d}{d\xi}\begin{pmatrix} u_2 \\ u_2' \end{pmatrix}\Big|_0 = \begin{pmatrix} u_2'(0) \\ u_2''(0) \end{pmatrix}$$

So the intersection can not exist, at most they can "touch".

So this monotonicity argument teaches us, that the set of c such that there exists a front is of the form:

either $[c^*, +\infty)$ or $(c^*, +\infty)$

c^* is necessarily finite by previous lemma

It suffies to prove that for $c = c^*$ there exists a trajectory between $\binom{1}{0} \to \binom{0}{0}$

A continuity argument works: if for some c the trajectory does not give a front, then it crosses the $\{u=0\}$ - axis. One can show that for $\tilde{c}$ close to c $(\tilde{c} > c)$ the orbit also crosses the $\{u=0\}$ -axis, which will lead to a contradiction. This continuity of an orbit w.r.t. c is non-trivial, but we omit the proof.

This finishes the proof for the general monostable case.

Rmk: Notice that for FKPP case

$$f'(0) = \sup_{u \in (0,1]} \frac{f(u)}{u} \quad , \text{thus}$$

$$c^* = 2\sqrt{f'(0)} \quad , \text{and}$$ item (iii) is also proven.

Next time we will prove the theorem for the bistable case, and, may be give a PDE proof of this theorem.

Lecture 20: We want to finish proving theorem:

$$(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1, \ u(+\infty) = 0, \quad u'(-\infty) = u'(+\infty) = 0 \end{cases}$$

Thm (existence of TW solutions)

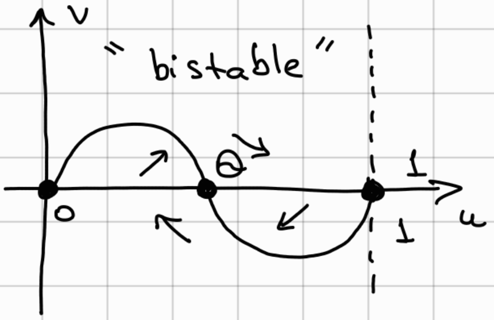
(i) In the bistable (and combustion) cases there exists a unique $c \in \mathbb{R}$ for which there exists a solution of $(TW)_\infty$.

Moreover,
- u is unique and decreasing;
- sign of c coincides with the sign of $\int_0^1 f(u)\,du$.

Proof (only sketch)

▶ Let $\int_0^1 f(u)\,du > 0$ (the other case is done similarly)

$u \in [0,1]$



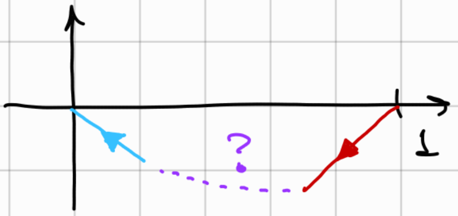
$$\begin{cases} u' = v \\ v' = -cv - f(u) \end{cases}$$

fixed points: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha = \{0, \theta, 1\}$

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4f'(\alpha)}}{2} \quad ; \quad r_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}$$

Rmk: $f'(0)$ and $f'(1)$ are negative $\Rightarrow$ $\lambda_\pm \in \mathbb{R}$ and, moreover, $\lambda_+ > 0$, $\lambda_- < 0$ thus both 0 and 1 are saddle points.

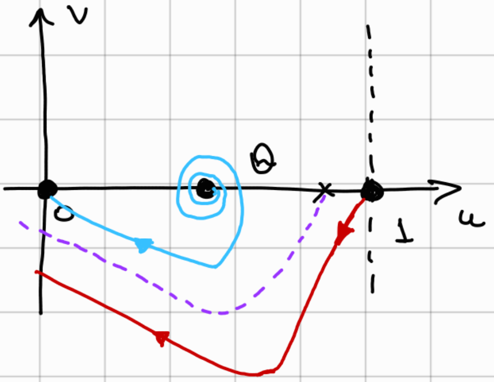


So the only way to have an orbit from $\begin{pmatrix}0\\0\end{pmatrix}$ to $\begin{pmatrix}1\\0\end{pmatrix}$ is when the unstable manifold (trajectory) from point $\begin{pmatrix}1\\0\end{pmatrix}$ coincides with the stable manifold of point $\begin{pmatrix}0\\0\end{pmatrix}$.
It is natural that this is a rare situation (despite the FKPP case where $\begin{pmatrix}0\\0\end{pmatrix}$ was node and locally all trajectories are attracted by $\begin{pmatrix}0\\0\end{pmatrix}$).

Idea: find two c s.t. we have:

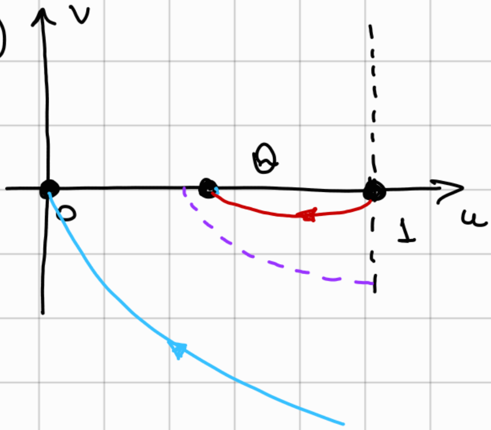
"blue" is above "red"

(I)



"blue" is below "red"

(II)



Then by continuity there exists c^* where "blue" and "red" intersect and, thus, coincide

(I) Take $c \le 0$ It can be proven that trajectory passing through point $\begin{pmatrix}\gamma\\0\end{pmatrix}$ will necesserily intersect the axis $u=0$ for $v<0$. This is a natural "barrier" between the "blue" and "red" orbits.
No proof. Moreover, the set of c with such property is open, and by monotonicity is, say $(-\infty, c_1)$.

(II) Take $c \gg 1$. Notice that the restriction of f on $[\theta, 1]$ is of monostable type, for at least for one c_2 the "red" orbit will go $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}\theta\\0\end{pmatrix}$ (and as a consequence for all $c > c_2$)

As a result there is a segment $[c_1, c_2]$, for which there exists an orbit from $\binom{1}{0} \mapsto \binom{0}{0}$. It remains to show that $[c_1, c_2]$ consists of 1 point. By contradiction, assume $c_1 < c_2$ such that the unstable "red" trajectory of $\binom{1}{0}$ converges to $\binom{0}{0}$. Let's call them T_1, T_2.

- As before, by monotonicity in c, T_1 is not above T_2.
- On the other hand the tangent vector for $T_{1,2}$ at point $\binom{0}{0}$ is $r_{1,2} = \binom{1}{\lambda_-^{1,2}}$

$$\lambda_-^{1,2} = \frac{-c_{1,2} - \sqrt{c_{1,2}^2 - 4f'(0)}}{2}$$

Notice that $\lambda_-^1 > \lambda_-^2$, which gives a contradiction (see picture below):

(!?) as T_1 is NOT above T_2

⌊ ∎

"PDE" proof of existence of TW solutions.

Step 1: Let $a \geq 1$ and consider

$$(TW)_a \quad \begin{cases} -u'' - cu' = f(u) & \text{in } (-a, a) \\ u(-a) = 1, \quad u(+a) = 0 \end{cases}$$

Proposition 1: $\forall a, c \ \exists! \ u = u_{a,c}$, $0 < u < 1$, $u' < 0$.

Proof:

⌈ ▶ (∃) $u \equiv 0$ - subsolution, $u \equiv 1$ - supersolution (e.g. Perron's method) } $\Rightarrow$ $\exists$ solution $0 < u < 1$

(!) Sliding method : take 2 solutions : u, v of $(TW)_a$

Let's prove that $u \le v$ and $v \le u$ (and thus, $u \equiv v$)

$\boxed{u \le v}$ Notice that $\forall h > 0$ $u(x+h)$ also satisfies the equation $-u'' - cu' = f(u)$, as the eq. is translation invariant.

Consider $u(x+h)$ for $0 < h < 2a$ and h being close to $2a$. Then on the interval $x \in [-a, a-h]$ $u(x+h) \le v(x)$ as $u(x+h)$ is close to $u(a) = 0$ and $v(x)$ is close to $v(-a) = 1$ (and are continuous in x)

Start decreasing h (that is moving the graph $u(x+h)$ to the right) and consider h_0 s.t.:

$$h_0 = \inf\{ h^* \in (0, 2a) : u(x+h) < v(x) \ \forall x \in [-a, a-h] \ \forall h \in (h^*, 2a)\}$$

That is the "first moment" when the graphs $u(x+h)$ and $v(x)$ touch, that is

$$\begin{cases} u(x+h_0) \le v(x) & \forall \ x \in [-a, a-h] \\ \exists \, x_0 \in [-a, a-h] : & u(x_0 + h_0) = v(x_0) \end{cases}$$

- If $h_0 = 0$, then $u \le v$ and this is what we want
- If $h_0 > 0$, then notice that $x_0 \ne -a$ as $u(-a + h_0) < 1 = v(-a)$. Also $x_0 \ne a - h_0$ as $u(a - h_0 + h_0) = u(a) = 0 < v(a - h_0)$.

So $\exists x_0 \in (-a, a-h) : u(x_0+h_0) = v(x_0)$

But this is a contradiction with the strong maximum principle as $u(x+h)$ is a subsolution and v is a solution, so they can not touch in an interior point of the domain. Thus, h_0 can not be positive.

$\boxed{v \leq u}$ Exchanging the positions of u and v in the previous argument, we get $v \leq u$.

Thus, we have proven the uniqueness.

$\boxed{u' < 0}$ Let's again use the sliding method, but now for $u(x+h)$ and $u(x)$.

Again for $h \approx 2a$ we have

$$u(x+h) < u(x).$$

Take

$$h_0 = \inf \{ h^* \in (0,2a) : \quad u(x+h) < u(x) \quad \forall x \in [-a, a-h] \quad \forall h \in (h^*, 2a) \}$$

- If $h_0 = 0$, then $\forall h \in (0,2a)$ $u(x+h) < u(x)$, and this gives $u' \leq 0$ (only non-strict inequality)

- If $h_0 > 0$, then by the same argument (strong maximum principle) we get a contradic.

Now let's show that, indeed, $u' < 0$ (strict ineq.)

Differentiate the eq. in $(TW)_a$:

$$-u''' - cu'' = f'(u) \cdot u'$$

Denote $v = u'$ and consider $f'(u)$ as known function:

Then: $\quad -v'' - cv' = g(\xi)\, v$

$V_0 \equiv 0$ is a solution of and $v = u' \leq 0$ is also a solution. By strong maximum principle, either $v \equiv 0$ or $v < 0$.

L As $u \not\equiv 0 \Rightarrow v \not\equiv 0 \Rightarrow v = u' < 0$. ■

We want now to fix c and take limit $a \to +\infty$. But the theorem says that only for some special c there exist a solution. Why this is happening? For many choices of c the solution will "run away" and in "almost all" points converge to 1 or 0:

or

So in the limit you get zero information as the solution converges to 1 or 0 (steady states that we already know)

"Pinning": let's restrict ourselves only to such solutions that have a prescribed value at 0:

Proposition 2: $\exists!$ c s.t. the corresponding u satisfies an extra condition $u_{a,c}(0) = \theta$, where θ is:

- bistable case: the unstable equil. $\theta \in (0,1)$
- ignition case: $\sup\{u<1 : f(u) = 0\}$
- monostable case: $\forall\, \theta \in (0,1)$

Proof

┌▶ For a moment assume no condition $u_{a,c}(0) = \theta$

Consider a mapping : $c \mapsto u_c$

It is decreasing and continuous.

Why decreasing?

- Take a solution u for some value c_1

Then it is a supersolution for $c_2 > c_1$ (due to sign $u' < 0$)

$$-u'' \underbrace{-u' c_2}_{>0} - f(u) > -u'' - u' c_1 - f(u) = 0$$

$$\Rightarrow \quad u_{c_2} < u_{c_1}$$

exercise

$$\begin{cases} \text{As } c \to +\infty & u_c(x) \to 0 \quad \text{in } (-a,a] \\ \text{As } c \to -\infty & u_c(x) \to 1 \quad \text{in } [-a,a) \end{cases}$$

└ All the above gives the unique c : $u_{a,c}(0) = \theta$

Let's prove an apriori bound on c from Prop.2 (to be able to get limit of c when $a \to \infty$)

Lemma : Let $m = \sup\limits_{s \in (0,1]} \frac{f(s)}{s}$.

$$\forall \delta > 0 \quad \exists A > 0 \quad \text{s.t.} \quad \forall a \geq A \quad c \leq 2\sqrt{m} + \delta.$$

Proof:

┌▶ Consider a problem:

$$(Z) \quad \begin{cases} -z'' - cz' - mz = 0 & \text{in } (-a,a) \\ z(-a) = 1, \quad z(a) = 0 \end{cases}$$

The solution u of $(TW)_a$ is a sub-solution of (Z)

$$mu = \sup_{v \in (0,1]} \frac{f(v)}{v} \cdot u \geqslant \frac{f(u)}{u} \cdot u = f(u)$$

$$-mu \leq -f(u)$$

Claim: the operator $\mathcal{L} = -\partial_{xx}^2 - c\partial_x - m$ satisfies the maximum principle (MP) in $(-a,a)$ for $c > 2\sqrt{m}$ provided a is large enough.

(no proof for a moment)

Assume by contradiction that $c > 2\sqrt{m} + \delta$

Then by claim the operator $\mathcal{L} = -\partial_{xx}^2 - c\partial_x - m$ satisfies the maximum principle, thus, for $w = u - z$ we have $Lw \leq 0$ and $w(-a) = w(a) = 0$ $\Rightarrow$ $w \leq 0$ $\Rightarrow$ $u \leq z$ for a large enough.

But we can find z explicitly.

Indeed,

$$z(x) = \frac{e^{r_+(x-a)} - e^{r_-(x-a)}}{e^{-2r_+a} - e^{-2r_-a}}, \text{ where}$$

r_+, r_- are the 2 real roots of:

$$r^2 + cr + m = 0$$

Notice that $z(0) = \dfrac{1}{e^{-r_+a} + e^{-r_-a}} \xrightarrow[a \to +\infty]{} 0$

and thus, $u(0) \to 0$, which is a contradiction with "pinning" condition $u(0) = \theta$. $\blacksquare$

Rmk: one can bound c from below:

consider $v(x) = 1 - u(-x)$

$$\begin{cases} -v'' + cv' = -f(1-v) \\ v(-a) = 1, \; v(a) = 0 \end{cases}$$

$$\Rightarrow \quad -c \le 2\sqrt{m'} + \delta \quad \text{where } m' = \sup_{s\in(0,1]}\left(-\frac{f(1-s)}{s}\right)$$

$$c \ge -2\sqrt{m'} - \delta$$

So if c is too negative, then we $u(0)$ will go to 1 and can not satisfy the "pinning" condition $u(0) = \theta$.

Step 2:

So we can pass to the limit $a \to +\infty$ and there exists a convergent subsequence $c_a \to c$, $u_a \to u$.

If u_a' and u_a'' are bounded then by Arzela-Ascoli theorem we can take a convergent subseq. and pass to the limit in the eq.

$$(*) \quad \begin{cases} -u'' - cu' = f(u) & \text{in } \mathbb{R}, \; u\in[0,1] \\ u(0) = \theta & u' \le 0 \end{cases}$$

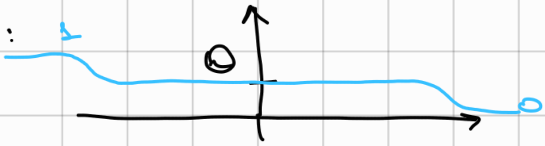
Monostable case We have shown that there exists at least 1 solution of $(*)$. Also we know that $u' \le 0$ and for $\xi \le 0$ $u \in [\theta, 1]$, so there exists a limit $u(-\infty) = u_0$

Also, $\left.\begin{matrix} u'(-\infty) = 0 \\ u''(-\infty) = 0 \end{matrix}\right) \Rightarrow u_0 : f(u_0) = 0$

This means that $u_0 = 1$.

Analogously, $u(+\infty) = 0$.

Bistable case The same reasoning does not work for the bistable case as there could happen:



that $u \equiv \theta$.

Lecture 21 : We finish "PDE" proof for existence of TW solutions for reaction-diffusion eq.

$$u_t = \Delta u + f(u)$$

and formulate the invasion / extinction criteria for monostable / bistable nonlinear.

- But first let's prove a version of (MP) that we left without proof in the previous class.

Lemma : Let $\mathcal{L} = -\frac{d^2}{dx^2} - c\frac{d}{dx} - m$ on $(-a,a)$

Here $c, m \in \mathbb{R}$. Assume $\boxed{c > 2\sqrt{m}}$.

If $\begin{cases} \mathcal{L}z \leq 0 \\ z(a) \leq 0 \\ z(-a) \leq 0 \end{cases} \Rightarrow z(x) \leq 0 \quad \forall x \in (-a,a)$

Proof :

▶ Trick (Liouville transform) : $\mathcal{L}z = 0$.

consider $z = e^{-c/2\, x}\varphi$

(this should kill the first order term in $\mathcal{L}$). Indeed,

$$\partial_x z = -\frac{c}{2} e^{-\frac{c}{2}x}\varphi(x) + e^{-c/2\, x}\varphi'(x)$$

$$\partial^2_{xx} z = \frac{c^2}{4} e^{-c/2\, x}\varphi - c e^{-c/2\, x}\varphi' + e^{-c/2\, x}\varphi''$$

$$\Rightarrow \mathcal{L}z = -\frac{c^2}{4} e^{-c/2\, x}\varphi \cancel{+ c e^{-c/2\, x}\varphi'} - e^{-c/2\, x}\varphi''$$

$$+ \frac{c^2}{2} e^{-c/2\, x}\varphi \cancel{- c e^{-c/2\, x}\varphi'} - m e^{-c/2\, x}\varphi =$$

$$= e^{-c/2\, x}\cdot\left[-\varphi'' + \varphi\left(\frac{c^2}{4} - m\right)\right]$$

Notice that $\mathrm{sign}(z) = \mathrm{sign}(\varphi)$.

If $\exists\, x_0 \in (-a,a)$: $\varphi(x_0) > 0$ (w.l.o.g. x_0 is argmax of φ), then $\varphi''(x_0) \leq 0$ and we have

$$-\varphi'' + \Big(\underbrace{\frac{c^2}{4}-m}_{\overset{\vee}{0}}\Big)\underset{\overset{\vee}{0}}{\varphi}\Big|_{x_0} > 0 \quad (!?) \quad Lz \leq 0.$$

⌞ $\Rightarrow \varphi \leq 0 \Rightarrow z \leq 0$ ■

- Travelling wave solutions satisfy the equation: $(TW)_\infty \begin{cases} -u'' - cu' = f(u) \\ u(-\infty) = 1,\ u(+\infty) = 0 \end{cases}$

Monostable case : we have shown that $\exists \lim\limits_{a\to+\infty} c_a = c$ such that $\exists$ solution of $(TW)_\infty$ with this c. Let's show that the solution of $(TW)_\infty$ $\exists$ for $[c, +\infty)$.

The following lemma is true only for monostable case (we use the fact that $f(u) > 0$, $u \in (0,1)$)

<u>Lemma</u> : If $\exists$ solution of $(TW)_\infty$ for c, then $\forall\, c_1 \geq c$ there also exists a solution of $(TW)_\infty$.

<u>Proof</u> :

⌜▶ Let u_c be a solution with c, then u_c is a supersolution for $c_1 > c$ and $u_c' < 0$. So is $u_c(\cdot + r)$, $r \in \mathbb{R}$

Introduce a finite-domain approximation:

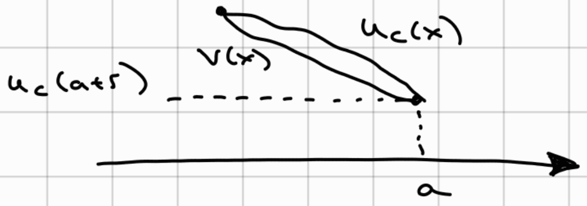
$$\begin{cases} -v'' - c_1 v' = f(v) \quad \text{in} \quad (-a,a) \\ v(-a) = u_c(-a+r) \\ v(+a) = u_c(a+r) \end{cases}$$

$u_c(\cdot + r)$ is a supersolution

$u(a+r)$ is a subsolution (it is constant)

Here we use $f(a) > 0$ $\forall a \in (0,1)$

$\Rightarrow \exists$ a solution $v(x)$:
$u_c(a+r) < v(x) < u_c(x+r)$



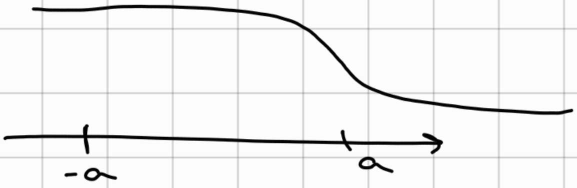
$$\Rightarrow \underbrace{\underset{v(a)}{\underset{\shortparallel}{u_c(a+r)}} < v(x) < \underset{v(-a)}{\underset{\shortparallel}{u_c(-a+r)}}}$$

Actually, the sliding method works! and only needs

$\Rightarrow$ v is unique and decreasing

By the same argument as before
$\exists!\, r$ s.t. $v(0)=\theta$

$r \gg 1$



$r \ll -1$

By continuity there exists r s.t. $v(0) = \theta$
Again tending $a \to \infty$ we get a limit
$v_a \to v$ and get a solution for c_1. ■

<u>Rmk</u>: the set of c for which there exists a solution of $(TW)_\infty$ is closed. Indeed, if we have a sequence of solutions (c_n, u_n) with $c_n \to c$ w.l.o.g. $u_n(0) = \theta$ so we can pass to the limit and get a solution of $(TW)_\infty$ with c.

Bistable case

We pass to the limit as

- c_a is bounded
- $u_a(0) = \theta$
- $u_a' \le 0$

$\Rightarrow \quad c_a \to c \, , \, u_a \to u \quad : \quad \begin{cases} -u'' - cu' = f(u) \\ u(0) = \theta \end{cases}$

So we need to show that $u \not\equiv \theta$.

It suffices to show that: $u_a'(0) \not\to 0$

Then $u'(0) < 0$, then the problem is reduced to "2 monostable" cases and $u(-\infty) = 1$ and $u(+\infty) = 0$.

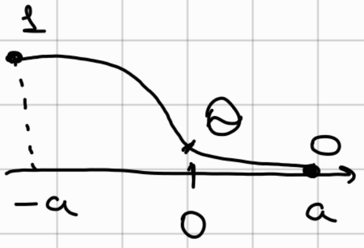
Let's show that $u_a'(0) \underset{a \to \infty}{\not\to} 0$.

Lemma : $\displaystyle\int_{-a}^{0} f(u_a(x))\, dx \ge \delta > 0 \quad \forall a \ge 1$.

Proof :

┌▶ Consider $\displaystyle\int_{-a}^{0} f(u_a) u_a' \, dx = F(\theta) - F(1)$

where $F(z) = \displaystyle\int_0^z f(u) du$



As $\|u_a'\|_{L^\infty} \le K$ and $f(u) > 0$, $u \in (\theta, 1)$

So $\delta' \le \left| \displaystyle\int_{-a}^{0} f(u_a) u_a' \right| \le \displaystyle\int_{-a}^{0} f(u) du \cdot \|u_a'\|_\infty$

$\Rightarrow \displaystyle\int_{-a}^{0} f(u) du \ge \frac{\delta'}{\|u_a'\|_\infty} \ge \frac{\delta'}{K}$ ■

└

$-u'' - cu' = f(u)$

Integrate this $\displaystyle\int_{-a}^{0}$: $\quad -u' \Big|_{-a}^{0} - cu \Big|_{-a}^{0} = \displaystyle\int_{-a}^{0} f(u) dx \ge \delta$

$$-u'(0) + \underbrace{u'(-a)}_{\le 0} - c\big[\underset{\theta}{\underset{\shortparallel}{u(0)}} - \underset{1}{\underset{\shortparallel}{u(-a)}}\big] \ge \delta$$

$$\Rightarrow \quad \boxed{c(1-\theta) - u'(0) \geq \delta}$$

We used the path from $-a$ to 0.

Let's use the other path from 0 to a:

$$-u'' - cu = f(u)$$

Integrate this $\int_0^a$: $\quad -u'\Big|_0^a - cu\Big|_0^a = \underset{\geq 0}{\int_0^a f(u(x))\,dx}$

$$-u'(a) + u'(0) + c\theta \leq 0$$

$$u'(0) \leq -c\theta + \underbrace{u'(a)}_{\leq 0} \leq -c\theta$$

Thus, $\boxed{u'(0) \leq -c\theta}$

Combining $\begin{cases} u'(0) \leq -\delta - c(1-\theta) \\ u'(0) \leq -c\theta . \end{cases}$

When $|c| < \dfrac{\delta}{(1-\theta)2}$, then $u'(0) \leq -\dfrac{\delta}{2}$
(c small)

Otherwise $u'(0) \leq -\dfrac{\delta\theta}{(1-\theta)2}$ again strictly negative

- Uniqueness of c^* for bistable case is a consequence of a sliding method (exercise) ■

Invasion, extinction and asymptotic speed of propagation

$$(*) \begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^N \\ u(0,x) = u_0(x), & u_0 \not\equiv 0, \ \underbrace{0 \le u_0 < 1}_{\text{for simplicity}} \end{cases}$$

Thm 1 (invasion for FKPP case)

Assume that $\lim\limits_{s \to 0^+} \frac{f(s)}{s^{1+\frac{2}{N}}} > 0$ (C1)

Then $\forall u_0(x)$ we have $u(t,x) \to 1$ as $t \to \infty$

Rmk 1: sometimes this is called "hair-trigger effect" – even small amount of species will invade everything (under the cond. (C1)). Cond. (C1) is sharp – there are counterexamples when (C1) is not true.

Thm 2 (extinction and invasion for bistable)

(i) (extinction) $\exists \delta > 0$ s.t. if $\int_{\mathbb{R}^N} (u_0 - \theta) < \delta$, then

$$u(t,x) \to 0 \quad \text{as} \quad t \to \infty \quad \forall x \in \mathbb{R}^N$$

(ii) (invasion) $\exists \eta > 0$, $R > 0$ s.t. if $u_0 \ge \theta + \eta$ on $\overline{B}_R$,

then $u(t,x) \to 1$ as $t \to \infty$ $\forall x \in \mathbb{R}^N$

Rmk 1: if there are not too many species then you have extinction, but if you have enough species on a big enough domain, you will have invasion.

Rmk 2: simpler version of (i): in $u_0 < \theta - \eta$ then $u \to 0$ (straighforward)

Rmk 3: Take $u_0 = \mathbb{1}_{B_R}$ for bistable case:

R small – extinction
R large – invasion

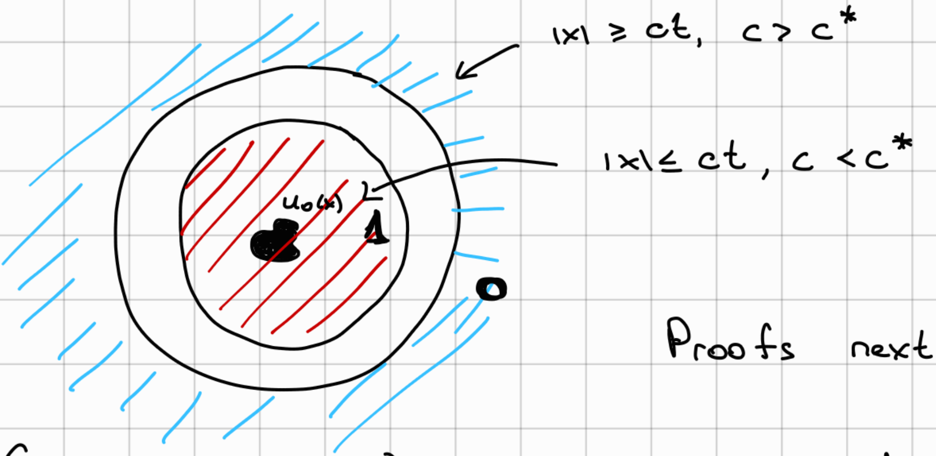
There is a threshold result : $\exists R^*$: $\forall R < R^*$ extinction and $R > R^*$ invasion.
[Zlatoš' 2006 – 1-dim ; Du & Matano' 2010 – N-dim]

Thm 3 (Principle of asymptotic speed of propagation) Assume that u_0 has compact support and that there is invasion. Then,

(1) $\forall c > c^*$, $$\lim_{t\to\infty} \left\{ \sup_{|x| \geq ct} u(t,x) \right\} = 0$$

(2) $\forall c < c^*$ $$\lim_{t\to\infty} \left\{ \sup_{|x| \leq ct} |1 - u(t,x)| \right\} = 0$$

Rmk: c^* – minimum speed of TW for monostable
c^* – is the unique speed of TW for bistable case



Proofs next time.

Rmk: $\begin{cases} u_t = d\Delta u + f(u), \\ u(0,x) = u_0(x). \end{cases}$ If one considers the eq. with diffusion coef. d then for Fisher - KPP case
$$c^* = 2\sqrt{d f'(0)}.$$
[change of variable $\sqrt{d}\cdot x$]

Lecture 22: Last time we formulated "extinction/survival" and ASP theorem. Let's prove them.

Proof of thm 1:

$\Gamma\blacktriangleright$ Instead of cond. (C1) we will use stronger condition $f'(0) > 0$

Step 1: subsolution with compact support

Consider an eigenvalue problem:

$$\begin{cases} -\Delta \varphi_R = \lambda_R \varphi_R & \text{in } B_R, \ \varphi_R > 0 \text{ in } B_R \\ \varphi_R = 0 & \text{on } \partial B_R \end{cases}$$

For R large enough and ε small enough $\varepsilon \varphi_R(x)$ is a subsolution of $-\Delta z = f(z)$ in B_R $\forall \varepsilon \leq \varepsilon_0$ due to: $\lambda_R = \frac{\lambda_1}{R^2} < f'(0)$

Here we extend φ_R by 0 outside B_R

Step 2: Take $\varepsilon > 0$ small enough s.t.

$$\varepsilon \varphi_R(x) < u(1, x) \quad \forall x \in \mathbb{R}^N$$

This can be done as by the maximum principle $u(t_0, x) > 0$ for $t_0 > 0$.

Let
$$w_t - \Delta w = f(w) \qquad (Eq)$$
$$w(0,x) = \begin{cases} \varepsilon \varphi_R(x) & \text{in } B_R \\ 0 & \text{if } |x| \geq R \end{cases}$$

Then: (a) w increases with t; (b) $w \leq 1$

Indeed, (a): consider an equation on w_t

We want to prove $w_t \geq 0$. Differenciate (Eq) w.r.t. t: $w_{tt} - \Delta w_t = f'(w) \cdot w_t$

Denote $v = w_t$ and $f'(w) = a(x,t)$ $\Rightarrow$

$$\begin{cases} v_t - \Delta v = a(x,t) v \\ v(0,x) = w_t(0,x) = \Delta w + f(w)(0,x) \geq 0 \end{cases}$$

as $\Delta w + f(w) \geq \Delta w + \lambda_R w = 0$
at$\uparrow$ point $(0,x)$

by the choice $w(0,x) = \begin{cases} \varepsilon \varphi_R \\ 0 \end{cases}$.

Thus, by the maximum principle :
$w_t(t,x) = v(t,x) \geq 0 \quad \forall\, t>0.$

(b) By a maximum principle,
$u_1 \equiv 1$ is a supersolution $\Rightarrow u(t,x) \leq 1$.

Thus, (a)+(b) $\Rightarrow$ $w(t,x)$ converges to $w_\infty(x)$ - a bounded function, and we have:

$$\Rightarrow \lim_{t \to \infty} u(t,x) \geq w_\infty(x) \qquad \forall\, x \in \mathbb{R}^N$$

Step 3 : $w_\infty(x)$ is the solution of the problem

$$(S) \quad \begin{cases} -\Delta w_\infty = f(w_\infty) \text{ in } \mathbb{R}^N \\ 0 < w_\infty \leq 1 \end{cases}$$

By Schauder estimates, we can prove that locally in any compact $K \subset [0,T] \times \mathbb{R}^N$ $w(t,x)$ and w_t, $w_{x_ix_j}$ are uniformly bounded, and by Arzela-Ascoli theorem there exists a convergent subsequence for all derivatives too. So we can write $(w_\infty)_t - \Delta w_\infty = f(w_\infty)$
As $w_\infty = w_\infty(x)$, we get $-\Delta w_\infty = f(w_\infty)$

For unbounded domains, we want to show: the only entire bounded solutions of (S) are $w_\infty \equiv 0$ and $w_\infty \equiv 1$.

In particular, in our case:

Proposition (Liouville-type theorem): $w_\infty \equiv 1$.

Proof: Sliding method

① $\inf_{\mathbb{R}^N} w_\infty > 0$ $\qquad 0 \le \varepsilon \le \varepsilon_0$

Take our subsolution  and start sliding (move everywhere)
Again by strong maximum principle $\varepsilon_0\varphi_R$ and w_∞ can not touch anywhere! Thus,
$w_\infty \ge \varepsilon_0$.

② $\inf_{x\in\mathbb{R}^N} w_\infty = 1$. By contradiction,

$$\exists\, x_0 : \quad w_\infty(x_0) = \min_{x\in\mathbb{R}^N} w_\infty(x) .$$

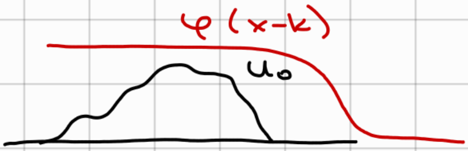
As $\qquad -\Delta w_\infty = f(w_\infty) > 0$, then
$\Delta w_\infty < 0$ at minimum(!?)

③ If $\exists\, x_n : |x_n| \to \infty$ and $w_\infty(x_n)$ converges to $\inf w_\infty < 1$, then we also have a contradiction.
Instead of sequence of points x_n take a sequence of functions $\tilde{w}_n(x)$
$\tilde{w}_n(x) = w_\infty(x - X_n)$
There exists a convergent subsequence which converges to $\tilde{w}_\infty(x)$. Moreover, $\tilde{w}_\infty(0) = \inf w_\infty$ and $-\Delta \tilde{w}_\infty = f(\tilde{w}_\infty)$ so by ② this can not happen. ∎

Proof of thm 3 :

① Upper bound on ASP. Let's remember a (TW) :

$$\begin{cases} -\varphi'' - c^*\varphi' = f(\varphi) \\ \varphi(-\infty) = 1, \quad \varphi(+\infty) = 0 \end{cases}$$



$0 \leq u_0 < 1$

We can translate TW solution $\varphi(x-k)$ for k large enough and make : $u_0(x) \leq \varphi(x-k)$

Claim : $\exists k > 0 \ \forall e \in S^{N-1} \ |e| = 1$ s.t. $u_0(x) \leq \varphi(x \cdot e - k)$

Thus, $\forall t > 0, x \in \mathbb{R}^N$ $\quad u(t,x) \leq \varphi(x \cdot e - c^* t - k)$

Taking e in the direction of x, we get :

$$u(t,x) \leq \varphi(|x| - c^* t - k)$$

Pick $c > c^*$:

$$\sup_{|x| \geq ct} u(t,x) \leq \varphi((c - c^*)t - k) \longrightarrow 0 \text{ as } t \to \infty$$

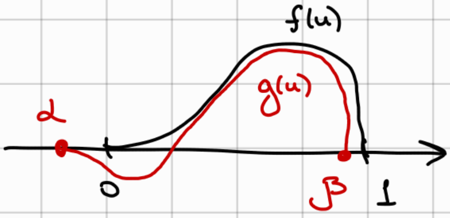
② Lower bound on ASP.

on $[0,1]$

Rmk : suppose f is either mono- or bistable.

Let g be a bistable on $[\alpha, \beta]$

$\alpha < 0 < \beta < 1$ s.t. $g \leq f$.



$$(1) \begin{cases} -u'' - c^* u' = f(u) \\ u(-\infty) = 1, \quad u(+\infty) = 0 \end{cases}$$

$$(2) \begin{cases} -v'' - c'v' = g(v) \\ v(-\infty) = \beta, \quad v(+\infty) = \alpha. \end{cases}$$

Then $c' < c^*$, and "$c' \to c^*$" as "$g \to f$"

In particular, when you approximate a monostable case by bistable cases, you get the smallest speed c^*.

Lecture 23: Comments for the last lecture:

Upper bound for FKPP case: can be done even more explicitly. Indeed, we have $f(u) < f'(0)u$ So we can consider a linear problem

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = f'(0)\tilde{u} \\ \tilde{u}(0,x) = u_0(x) \in [0,1] \end{cases}$$ - compactly supp on B_R

and solve it explicitly:

$$\tilde{u}(t,x) = \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} u_0(y)\, e^{-\frac{|x-y|^2}{4t}} dy$$

So it is clear that if $\tilde{u}(t,x) \to 0$, then the solution of the non-linear problem also $u(t,x) \to 0$.

But for $|x| \geq 2\sqrt{f'(0)}\,t$, we have $\tilde{u}(t,x) \to 0$.

Indeed, $|x-y|^2 = |x|^2 - 2xy + |y|^2 \geq |x|^2 - 2|x|R - R^2$

Then $\frac{|x-y|^2}{4t} \geq \frac{4 f'(0)\cdot t^2 - 2\cdot 2\sqrt{f'(0)}\,t - R^2}{4t} = f'(0)t + O(1)$

and $e^{f'(0)t - \frac{|x-y|^2}{4t}} \leq e^{O(1)}$ - bounded $\Rightarrow$

$$\tilde{u}(t,x) \sim \frac{\text{const}}{\sqrt{t}} \longrightarrow 0$$

We even see that the front is "to the left" of $x = 2\sqrt{f'(0)}\,t$. In fact, there is a logarithmic shift: $|x| = c^* t - \frac{3}{2\lambda^*}\ln t + C$ for λ^* explicit (in this case $= c^*/2$)

Lemma: Let f be of monostable or bistable and $u_0 \in [0,1]$. Then up to a subsequence the solution of $u_t - \Delta u = f(u)$ converges as $t \to +\infty$ to a stationary state: $-\Delta u_\infty = f(u_\infty)$

Rmk 1: There could exist other stationary states (not constants) apart from 0 and 1 (and θ) for the bistable case (e.g. in bounded domains)

Simple example: $f(u) = u - \theta$ in $[\theta - \delta, \theta + \delta]$

then $u = \theta + \delta \cos x$ solves $-u''(x) = f(u(x))$, $x \in [-\pi, \pi]$

In our example we will only encounter $u \equiv 0$, $u \equiv 1$ as a possible attracting stationary states.

Rmk 2: To thm 1: the "hair-trigger" effect for monostable nonlinearity can dissappear for $x \in \Omega$ - bounded domain with Dirichlet b.c. $u|_{\partial\Omega} = 0$ ("unfriendly" boundary)

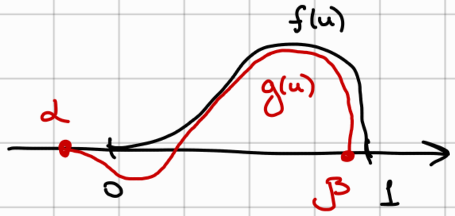
F.e. if the boundary "is close" to any interior point 

② Lower bound on ASP.

Rmk: suppose f is either mono- or bistable on $[0,1]$.

Let g be a bistable on $[\alpha, \beta]$

$\alpha < 0 < \beta < 1$ s.t. $g \leq f$.



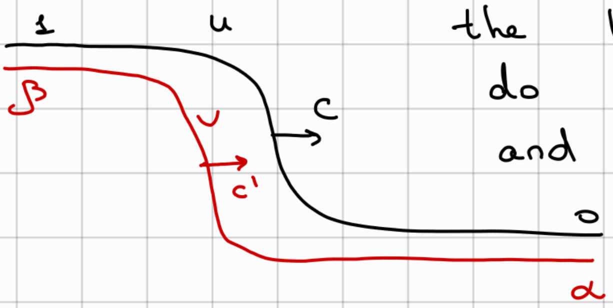
$$(1) \begin{cases} -u'' - c^* u' = f(u) \\ u(-\infty) = 1, \quad u(+\infty) = 0 \end{cases}$$

$$(2) \begin{cases} -v'' - c' v' = g(v) \\ v(-\infty) = \beta, \quad v(+\infty) = \alpha. \end{cases}$$

Then $c' < c^*$, and "$c' \to c^*$" as "$g \to f$"

In particular, when you approximate a monostable case by bistable cases, you get the smallest speed c^*.

Indeed, consider u and v : "slide" v to the left s.t. u and v do not intersect (v below u) and start push v back $v(x+h)$



Consider $h_0 := \inf\{ h_1 \in \mathbb{R} : v(x+h) < u(x), x \in \mathbb{R} \ \forall h > h_1 \}$

Then $\exists x_0 \in \mathbb{R} : \quad v(x_0+h_0) = u(x_0)$

If $c' \geq c^*$, then v is a subsolution for (1)

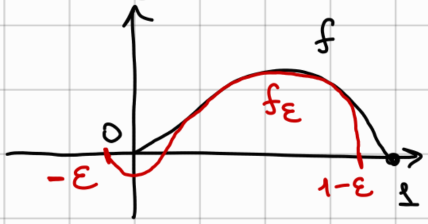
Indeed,

$$-v'' - c^* v' - f(v) \leq -v'' - c'v' - g(v) = 0$$

Thus, $v(x+h_0) \leq u(x)$ and $\exists x_0 : v(x_0+h_0) \leq u(x_0)$ and this is a contradiction by strong maximum principle.

In particular, for monostable case $c' < c^*$ – the minimal value of speeds.

So let's change f. We will approximate f by f_ε : $f_\varepsilon \leq f$ and $f_\varepsilon : [-\varepsilon, 1-\varepsilon] \to \mathbb{R}$



Consider a TW accosiated with f_ε:

$$-\varphi_\varepsilon'' - c_\varepsilon \varphi_\varepsilon' = f_\varepsilon(\varphi_\varepsilon) \text{ in } \mathbb{R}$$

$$\varphi_\varepsilon(-\infty) = 1-\varepsilon, \quad \varphi_\varepsilon(+\infty) = -\varepsilon$$

Moreover, $c_\varepsilon \leq c^*$ and $c_\varepsilon \to c^*$ when $f_\varepsilon \to f$.

Fix $c < \gamma < c_1 < c^*$ and take ε small enough s.t. $c_1 < c_\varepsilon < c^*$

Claim: $v = \varphi_\varepsilon(|x| - \gamma t)$ is a subsolution of $u_t = \Delta u + f(u)$ in $\mathbb{R}^N \setminus B_R$ for R large enough.

Proof:

▶ Indeed, $v_t - \Delta v - f(v) = -\gamma\varphi_\varepsilon' - \varphi_\varepsilon'' - \frac{N-1}{|x|}\varphi_\varepsilon' - f(\varphi_\varepsilon) \le$

$$\le -\gamma\varphi_\varepsilon' \underbrace{-\varphi_\varepsilon''}_{=c_\varepsilon\varphi_\varepsilon'} - \frac{N-1}{|x|}\varphi_\varepsilon' \underline{-f_\varepsilon(\varphi_\varepsilon)} =$$

$$= \left(\underset{\overset{\vee}{0}}{c_\varepsilon - \gamma} - \frac{N-1}{|x|}\right)\underset{\overset{\wedge}{0}}{\varphi_\varepsilon'} < 0$$

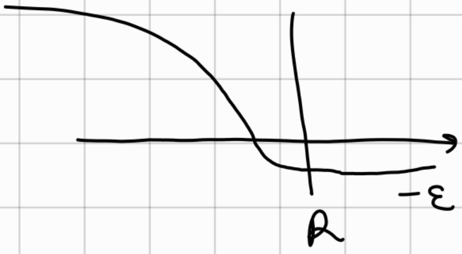
$\nearrow$ is small for $|x| > R >> 1$ ■

- Same is true for $\varphi_\varepsilon(|x| - \gamma t + k)$ for $\forall$ translation k as the eq. is translation invariant

- We have "invasion": $\forall \varepsilon > 0\ \exists T > 0$ s.t.

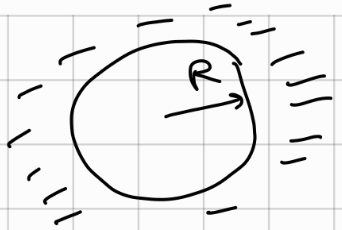
$$u(t,x) \geq 1 - \varepsilon \quad \forall x \in \overline{B_R} \quad \forall t \geq T$$

- Choose K large enough s.t. $|x| \geq R$ $\quad \varphi_\varepsilon(|x| + K) < 0$



Compare $u(t+T, x)$ and $\varphi_\varepsilon(|x| - \gamma t + k)$ in $\mathbb{R}^N \setminus \overline{B_R}$.

On ∂B_R $\quad u(t+T, x) \geq 1 - \varepsilon \geq \varphi_\varepsilon(\ldots)$



So outside B_R initially $u(T,x) \geq \varphi_\varepsilon(|x|+k)$ and $\forall t$ on ∂B_R $u(t+T,x) \geq \varphi_\varepsilon(\dots)$

So by MP this is true for all times in $\mathbb{R}^N \setminus \overline{B_R}$

$$u(t+T,x) \geq \varphi_\varepsilon(|x|-\gamma t+k)$$

But also is true inside B_R as

$$u(t+T,x) \geq 1-\varepsilon \geq \varphi_\varepsilon(\dots) \quad \forall t$$

$\Rightarrow \forall x \in \mathbb{R}^N \quad u(t+T,x) \geq \varphi_\varepsilon(|x|-\gamma t+k)$

Take $|x| \leq ct$, $c<\gamma<c^*$, we have

φ_ε is decreasing

$$u(t,x) \geq \varphi_\varepsilon(ct-\gamma t+\gamma T+k) =$$

$$= \varphi_\varepsilon((c-\gamma)t+\underbrace{\gamma T+k}_{\text{some shift}}) \longrightarrow 1-\varepsilon \text{ as } c-\gamma<0$$

Thus, for $c<c^*$

$$\underline{\lim}_{t\to\infty}\left\{\inf_{|x|\leq ct} u(t,x)\right\} \geq 1-\varepsilon \quad \forall \varepsilon>0.$$

As it is true for $\forall \varepsilon>0 \Rightarrow \geq 1 \Rightarrow =1.$ ∎

Proof of thm 2 (about bistable eq.)

Rmk 1: let f be of bistable type: 

and u_0 - initial data.

- If $0 \leq u_0 \leq\not\equiv \theta \Rightarrow u \to 0$ uniformly
- If $\theta \leq\not\equiv u_0 \leq 1 \Rightarrow u \to 1$ uniformly

Indeed, if $\theta \leq u_0 \leq 1 \Rightarrow \theta \leq u \leq 1$ by comparison prin.

Then we are in a monostable case! Thus, by thm 1 (as $u \not\equiv \theta$) we obtain $u \to 1$. Analogously, for $0 \leq u_0 \leq \theta$.

The question of interest is what happens if somewhere $u_0 > \Theta$ and somewhere $u_0 < \Theta$.

(ii) Let's prove an "invasion" result:

$\exists \eta > 0$, $R > 0$ s.t. if $u_0 \geq \Theta + \eta$ on $\bar{B}_R$, then $u(t,x) \to 1$ as $t \to \infty$ $\forall x \in \mathbb{R}^N$

We will follow this scheme (which we already have seen for thm 1)

Step 1 : construct a subsolution $\underline{u}(x)$ with compact supp.

Step 2 : take a solution v: $v_t - \Delta v = f(v)$, $v(0,x) = \underline{u}(x)$. prove that $\partial_t v \geq 0$, $v \leq 1$ $\Rightarrow$ converges $v(t,x) \to \underline{v}(x)$ – a stationary solution $-\Delta \underline{v} = f(\underline{v})$

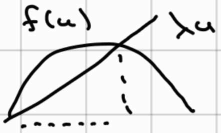
Step 3 : $\underline{v}(x) \equiv 1$. As a consequence,

$$1 \geq \lim_{t \to \infty} u(t,x) \geq \lim_{t \to \infty} v(t,x) = \underline{v}(x) \equiv 1.$$

$$\Rightarrow \quad u(t,x) \to 1 \quad \text{as} \quad t \to \infty.$$

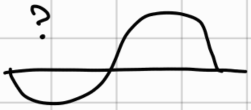
Let's start:

Step 1 : to construct a subsolution in thm 1 we used the estimate $\lambda u \leq f(u)$ for small u and small λ.



So we could consider a linear problem: $-\Delta \varphi = \lambda \varphi$

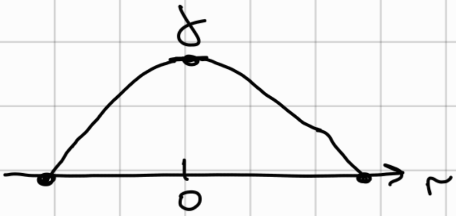
Here we can not the same



So how to construct a subsolution?

Consider: (first, 1-dim case) $p = p(r)$, $r \geq 0$

$$\begin{cases} -p''(r) = f(p(r)) \\ p(0) = \gamma > \Theta \\ p'(0) = 0 \end{cases}$$

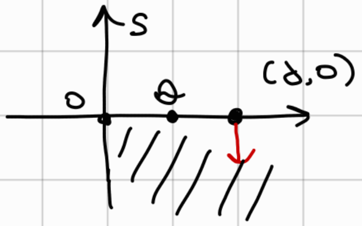


We are looking for a "radially symmetric" (even) function $p(r)$ that is positive and $p(R) = 0$ at some $R > 0$.

Write this equation as a system:

$$\begin{cases} p' = s & p(0) = \gamma \\ s' = -f(p) & s(0) = 0 \end{cases}$$

and draw a phase portrait:



$\gamma > \theta$

$-f(\gamma) < 0$ as ✓

So for some time (p,s) will be in domain $\{p>0\} \cap \{s<0\}$

p is decreasing and there exist 2 possib.

1)

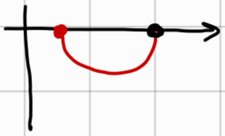
$\exists R > 0 : \quad p(R) = 0$

$p'(R) < 0$

and $\forall\ r \in [0, R) \quad p(R) > 0$

or

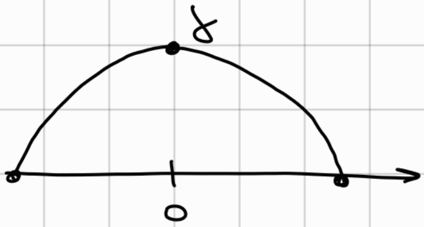
2)



$\exists R > 0 : \quad p'(R) = 0$

$p(R) \in [0, \theta]$

If Situation 1) happens, then we have found our subsolution:



$$\underline{u}(x) = \begin{cases} p(|x|), & |x| \leq R \\ 0, & |x| \geq R \end{cases}$$

Why it is a subsolution?

- $\underline{u}(x) \leq \gamma \leq u_0(x)$ on B_R if we take $\eta = \gamma - \theta$

  $\underline{u}(x) = 0 \leq u_0(x)$ outside B_R

- $\partial_t \underline{u} - \underline{u}'' = f(\underline{u})$ inside and outside B_R and  has a correct change of derivatives to be a "generalised" subsolution in weak sense.

Let's show that for γ close enough to 1 Situation 2) can not happen.

Take $-p'' = f(p)$, multiply by p' and integrate:

$$0 = \int_0^R (p''p' + f(p)p')\,dr = \frac{(p')^2}{2}\Big|_0^R + F(p(r))\Big|_0^R$$

where $F(p(r)) = \int_0^R f(p(s))\,p'(s)\,ds$

Thus, $0 = 0 + \int_{p(R)}^{\gamma} f(u)\,du \underset{\text{by the choice of } \gamma}{\overset{}{>}} 0$ (!?)

<u>Step 2</u>: prove that $\partial_t \underline{u} \geq 0$, $\underline{u} \leq 1 \Rightarrow$ converges

$\underline{u}(t,x) \to \underline{V}(x)$ - a stationary solution

$-\Delta \underline{V} = f(\underline{V})$

Is analogous to the proof of thm 1.

Indeed, consider a solution v:

$$\begin{cases} v_t - v'' = f(v) \\ v(0,x) = \underline{u}(x) \end{cases}$$

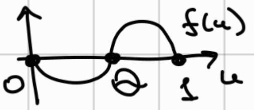
<u>Observations</u>:
- $v(t,x) \geq \underline{u}(x)$ [as $\underline{u}(x)$ is a subsolution]
- $\partial_t v \geq 0$ - the same as before
- $v \leq 1$ as 1 is a super solution

$\Rightarrow v(t,x) \to \underline{V}(x)$: $-\underline{V}'' = f(\underline{V})$

<u>Step 3</u>: $\underline{V}(x) \equiv 1$. We will prove next time.

Lecture 24: LAST LECTURE!

We are proving the "extinction / invasion" thm for reaction-diffusion eq. with bi-stable nonlinearity



$$(*) \quad \begin{cases} u_t = \Delta u + f(u) \\ u(0,x) = u_0(x) \text{ - compactly supp} \end{cases}$$

Thm 2 (extinction and invasion for bistable)

(i) (extinction) $\exists\, \delta > 0$ s.t. if $\int_{\mathbb{R}^N} (u_0 - \theta)_+ < \delta$, then

$$u(t,x) \to 0 \quad \text{as} \quad t \to \infty \quad \forall\, x \in \mathbb{R}^N$$

(ii) (invasion) $\exists\, \eta > 0$, $R > 0$ s.t. if $u_0 \geq \theta + \eta$ on $\overline{B}_R$,

then $u(t,x) \to 1$ as $t \to \infty$ $\forall\, x \in \mathbb{R}^N$

Proof:

▶ (ii) First, let's prove "invasion" part.

We will follow the scheme (that we already have seen for thm 1)

Step 1: construct a subsolution $\underline{u}(x)$ with compact supp.

Note that $\underline{u}(x)$ depends only on x!

Step 2: take a solution v: $v_t - \Delta v = f(v)$, $v(0,x) = \underline{u}(x)$.

Prove that v is a subsolution of $(*)$.

Moreover, $\partial_t v \geq 0$, $v \leq 1$ $\Rightarrow$ converges

$v(t,x) \to \underline{v}(x)$ - a positive stationary solution $-\Delta \underline{v} = f(\underline{v})$

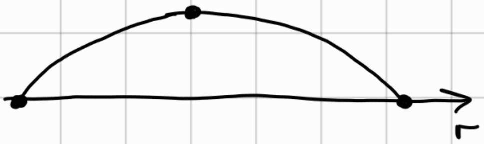
Step 3: $\underline{v}(x) \equiv 1$. As a consequence,

$$1 \geq \lim_{t \to \infty} u(t,x) \geq \lim_{t \to \infty} v(t,x) = \underline{v}(x) \equiv 1.$$

$$\Rightarrow \quad u(t,x) \longrightarrow 1 \quad \text{as} \quad t \to \infty.$$

Last time we already did step 1 for space dimension $N=1$. We looked for a radial function $p=p(r)$:

$$(1)\begin{cases} -p''(r) = f(p(r)) \\ p(0) = \gamma > 0 \\ p'(0) = 0 \end{cases}$$



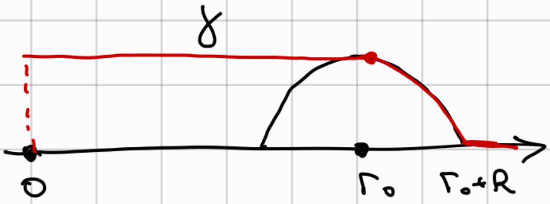
and chose γ s.t. $\exists R_0 > 0$: $p(r) > 0, \quad r \in [0, R_0)$
$p(R_0) = 0$

Before going to steps 2,3, let's generalize this construction to any space dimension, $N \geq 2$. Instead of ODE (1) we need to consider

$$(2)\begin{cases} -p''(r) + \frac{N-1}{r} p' = f(p(r)) \\ p(r_0) = \gamma \\ p'(r_0) = 0 \end{cases}$$

for some $r_0 > 0$.

If r_0 is big enough then $\frac{N-1}{r_0}$ is small and one can consider system (2) as "a small perturbation" of system (1). By continuity of solutions of ODE, we conclude that $\exists R$: $p(r_0+R) = 0$
$\forall r \in [0,R)$ $p(r_0+r) > 0$.



So we can define

$$\underline{u}(x) = \begin{cases} \gamma, & |x| < r_0 \\ p(|x|), & |x| \in (r_0, r_0+R) \\ 0, & |x| \geq r_0+R \end{cases}$$

Step 2: consider $v_t - \Delta v = f(v)$
$v(0,x) = \underline{u}(x)$.

By maximum principle, $v(0,x) \leq u(0,x)$
$\Rightarrow v(t,x) \leq u(t,x) \quad \forall t > 0$

Is analogous to the proof of thm 1.

Observations : • $v(t,x) \geq \underline{u}(x)$
[as $\underline{u}(x)$ is a subsolution]

- $\partial_t v \geq 0$ – the same as before
- $v \leq 1$ as 1 is a super solution

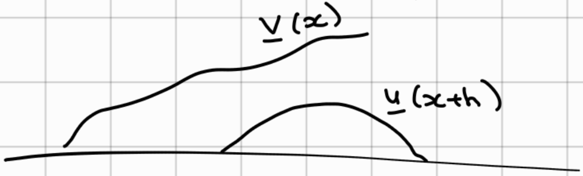
$\Rightarrow$ $v(t,x) \to \underline{v}(x)$: $-\underline{v}'' = f(\underline{v})$

Step 3 : $\underline{v}(x) \equiv 1$. The same as in thm 1:

$\boxed{3.1}$ $\inf \underline{v}(x) \geq \gamma = \sup \underline{u}(x)$.

The proof is by "sliding" method.

$\underline{u}(x+h)$ is a subsolution of $\underline{v}(x)$ $\forall h \in \mathbb{R}$ (for the equation). If $\forall h$ $\underline{u}(x+h) \leq \underline{v}(x)$, then $\underline{v}(x) \geq \gamma$ and we win. By contradiction, consider $\underline{u}(x+h), h=0$ and start "sliding" for $h>0$ and $h<0$.



Take "the first h_0" such that you can not move to the right or to the left:

$$h_0 = \min(\sup A, \inf B), \text{ where}$$

$$A = \{ h>0 : \forall \tilde{h} \in [0,h) \quad \underline{u}(x+h) \leq \underline{v}(x) \}$$
$$B = \{ h<0 : \forall \tilde{h} \in (h,0] \quad \underline{u}(x+h) \leq \underline{v}(x) \}$$

Then for this h_0 we have a touching point $x_0 \in \mathbb{R}$ between $\underline{u}(x_0+h_0) = \underline{v}(x_0) > 0$. By a strong maximum principle $\underline{u}(x+h_0) \equiv v(x)$ which is a contradiction, as $\underline{u}(x+h_0)$ has a compact supp, and $v(x)$ does not.

$\boxed{3.2}$ $\inf \underline{v}(x) = 1$. By contradiction, either $\exists x_0 \in \mathbb{R}$: $\underline{v}(x_0) = \min\limits_{x \in \mathbb{R}} \underline{v}(x)$

As $-\Delta \underline{v} = f(\underline{v}) > 0$ $(\underline{v} \geq \alpha > \theta)$
then $\Delta \underline{v} < 0$ at minimum (!?)
And similar as before we treat the case when $\exists x_n$: $|x_n| \to \infty$ and $\underline{v}(x_n) \to \inf\limits_{x \in \mathbb{R}} \underline{v}(x)$

The proof of (ii) is finished.

(i) We will prove a little bit weaker version

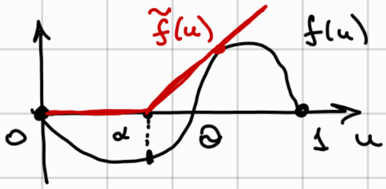
$\forall 0 < \alpha < \theta$ $\exists \delta > 0$ s.t. if $\int\limits_{\mathbb{R}^N} (u_0 - \alpha)_+ < \delta$, then

$$u(t,x) \to 0 \quad \text{as} \quad t \to \infty \quad \forall x \in \mathbb{R}^N$$

Rmk: in the proof we will see that we can take $\delta = (\theta - \alpha) \cdot$ const

thus, at least our approach is valid for $\forall \alpha < \theta$, but not $\alpha = \theta$.
Any ideas on the proof for $\alpha = \theta$ are welcome!

Now our goal is to construct a supersolution that tends to 0.
To do this, let's construct $\tilde{f}(u) \geq f(u)$



$$\tilde{f}(u) = \begin{cases} 0, & u \in [0, \alpha] \\ s \cdot u, & u \in [\alpha, 1], \end{cases} \quad \text{where}$$

$$s = \sup_{u \in [0,1]} \frac{f(u)}{u - \alpha} \quad \text{(see Fig.)}$$

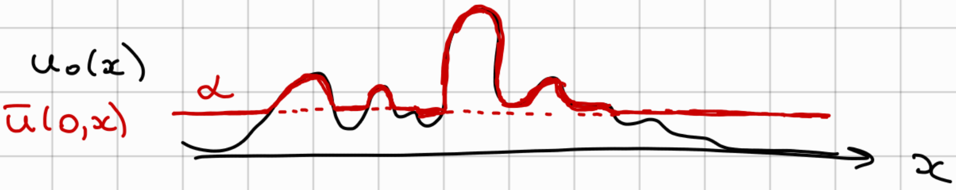
- Now consider the following problem:

$$\begin{cases} v_t = \Delta v + \tilde{f}(v) \\ v(0,x) = (u_0(x) - \alpha)_+ \end{cases}$$

Note that $v(t,x) \geq 0$, thus $v_+ = v$.
The function $\bar{u}(t,x) = v(t,x) + \alpha$ is a supersolution to (*). Indeed

$$\bar{u}_t - \Delta \bar{u} - f(\bar{u}) \geq v_t - \Delta v - \underbrace{sv}_{\tilde{f}(v)} = 0$$

$$\bar{u}(0,x) = (u_0(x) - \alpha)_+ + \alpha \geq u_0(x)$$



And we can write explicitly the answer:

$$\bar{u}(t,x) = \frac{e^{st}}{2\sqrt{\pi t}} \int_{\mathbb{R}^N} e^{-\frac{|x-\xi|^2}{4t}} (u_0 - \alpha)_+ \, d\xi + \alpha$$

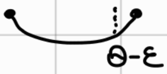
$$\leq \frac{e^{st}}{2\sqrt{\pi t}} \int_{\mathbb{R}^N} (u_0 - \alpha)_+ \, d\xi + \alpha$$

Take $t = \frac{1}{s}$ $\Rightarrow$ $\bar{u}(\frac{1}{s}, x) \leq \frac{e\sqrt{s}}{2\sqrt{\pi}} \int_{\mathbb{R}^N} (u_0 - \alpha)_+ \, d\xi + \alpha$

If $\bar{u}(\frac{1}{s}, x) < \theta - \varepsilon$ (for some ε) $\Rightarrow$ $\bar{u}(t,x) \xrightarrow[t \to \infty]{} 0$

(By comparing with solution to ODE:

$$\begin{cases} w_t = f(w) \\ w(0) = \theta - \varepsilon \end{cases}$$



$\Rightarrow w \to 0$)

Thus, if $\int_{\mathbb{R}^N} (u_0-\alpha)_+ \, dz < (\theta - \alpha) \frac{2\sqrt{\pi}}{e\sqrt{s}} \Rightarrow$

$$u(t,x) \le \bar{u}(t,x) \longrightarrow 0 \quad \text{as } t \to \infty \quad \forall x \in \mathbb{R}^N.$$

∎

More sharp result:

(KPP'1937): $f(u) = u(1-u)$, $c^* = 2\sqrt{f'(0)} = 2$

Take $u_0(x) = \mathbb{1}_{(-\infty, 0]}$.

There exists a function $\sigma^\infty(t) = 2t + o(t)$ s.t.

$$\lim u(t, x + \sigma^\infty(t)) = U_{c^*}(x)$$

where $U_{c^*}(x)$ is a TW solution, namely

$$\begin{cases} -c^* U_{c^*}' - U_{c^*}'' = f(U_{c^*}) \\ U(-\infty) = 1, \quad U(+\infty) = 0 \end{cases}$$

Bramson'1983
Roquejoffre
Hamel, Nolen
Ryzhik } '2013

Proved a sharp position of the front:

$$\sigma_\infty(t) = 2t - \frac{3}{2}\ln t - x_\infty + \underset{t\to\infty}{o(1)}$$

We will stop here. Good luck on exam!

☺