

# Proofs from the BOOK

## DMAT PUC-Rio, 2024.1

### Homework 1: Warming up... Deadline: 01.04.2024, 23:59

Больше задач, хороших и разных!  
Winston Churchill (1918)

1. Prove that if  $n \geq 2k$ , then at least one of the numbers  $n, n - 1, \dots, n - k + 1$  has a prime divisor  $p$  greater than  $k$ .

*Equivalent formulation:* prove that the binomial coefficient  $\binom{n}{k}$  always has a prime factor  $p > k$ .

*Hint:* follow the steps from the proof of Bertrand's postulate.

2. Consider the Buffon's needle problem for a "long" needle. Find the probability of the intersection of the needle of length  $l$  with the parallel grid of lines (distance between neighbouring lines is  $d \leq l$ ). Show that for  $l = d$ , the answer coincides with the answer for "short" needle ( $d \geq l$ ). What happens to this probability as  $l \rightarrow \infty$ ?

3. A family has two kids.

- A) It is known that one of them is a boy. What is the probability that the other child is also a boy?
- B) It is known that one of them is a boy born on Friday. What is the probability that the other child is also a boy?
- B) It is known that one of the kids is a boy born between 6 pm and 7 pm on Friday. What is the probability that the other child is also a boy?

4. The rules of the Moscow subway restrict the size  $L = 150$  cm of a luggage (rectangular parallelepiped) that can be taken along, where  $L$  is the sum of the parallelepiped's measurements:  $L = \text{length} + \text{width} + \text{height}$  (see <https://www.mosmetro.ru/info/>, Section 2.10.1). May one cheat by putting a box of a larger size inside a smaller one?

5. (Arnold's mathematical trivium) Find the mathematical expectation of the area of the orthogonal projection of a cube in  $\mathbb{R}^3$  with edge length 1 on a random plane (with an isotropically distributed random direction of projection).

*Hint:* try to find the BOOK proof arguing as in Buffon's problem.

6. Given a cone (a mountain), throw a loop on it and pull it tight (to climb). Clearly, if the cone is sufficiently acute, it will work, and if it is considerably obtuse, the loop will slide away. What is the borderline angle?

7. How to be 1/3 Spanish? Three questions (almost equivalent).

- A) A river has a flux of water equal to one (for instance, 1 cubic meter of water per second). Let us call a device that can split the flow into two equal parts a simple divider. So by applying such a simple divider to the river, one can fork it into two smaller streams with fluxes 1/2 each. Can one use a finite number of such simple dividers to separate 1/3 of the flow?
- B) Three thieves would like to split their loot, item by item, via a lottery with equal probability 1/3 of winning for each of them with the help of a standard "heads-or-tails" coin. How should they organize their lottery?
- B) "Can one be 1/3 Spanish?". When only one of two parents is Spanish, it makes their child 1/2 Spanish. If only one of his or her four grand-parents is Spanish, the child is 1/4 Spanish. So can one be 1/3 Spanish?

*Hint:* think out of the box!

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